Problem Set 1

Hi. Welcome to this problem set. Our goal here is going to be to work through some concrete examples to make sure you understand the fundamentals that we covered in the first two lectures. For starters, you may want to read through the notes for lectures 1 and 2, and/or the fancy typed notes. Or you can get right to the problems and go back to the notes later. If for some reason you are feeling rusty about complex numbers, now would be a good time to brush up on those. One way to do that is by going through the questions on the old Physics 200 tutorial that I've posted in the reading section for Tuesday's lecture.

The qubit

First, we'll consider a few questions about the simple quantum system known as a **qubit**. Not to be confused with *qbert*.

As we discussed in class, a qubit refers to any quantum system whose states are vectors in a two-dimensional Hilbert space.[1] An important example is the physical system describing the spin states of an electron. As we did in class, we'll call the basis state with $S_z = \hbar/2$ $|\uparrow\rangle$ and the basis state with $S_z = -\hbar/2$ $|\downarrow\rangle$. So we have

$$\langle\uparrow|\uparrow\rangle = 1 \qquad \langle\uparrow|\downarrow\rangle = 0 \qquad \langle\downarrow|\uparrow\rangle = 0 \qquad \langle\downarrow|\downarrow\rangle = 1$$

A general state for the electron's spin can then be written as

$$|\Psi\rangle = z_1|\uparrow\rangle + z_2|\downarrow\rangle\ . \tag{1}$$

Problem 1 (Webwork)

Consider the states

$$\begin{aligned}|A\rangle &= \frac{1}{2}|\uparrow\rangle - i\frac{\sqrt{3}}{2}|\downarrow\rangle \\ |B\rangle &= \frac{i}{2}|\uparrow\rangle + \frac{\sqrt{3}}{2}|\downarrow\rangle\end{aligned}$$

[1] By contrast, qbert (or more precisely, Q*bert), refers to a fictional character originally appearing in video games from the 1980s who typically hops around on a pyramid made of 28 cubes, trying to change every cube to a target color while avoiding enemies. But that isn't relevant for this course.

What are $\langle A|A\rangle$, $\langle A|B\rangle$, $\langle B|A\rangle$, $\langle B|B\rangle$? Complete this one in Webwork to check your answers. To access Webwork for the first time, follow the link in the Canvas page for this course. You should have a close look at the properties of the inner product in the typed class notes for lecture 1, particularly the third property.

We said that vectors related through multiplication by a nonzero complex number represent the same state. So a state (1) defined by coefficients (z_1, z_2) is the same as a state defined by coefficients (wz_1, wz_2). By such a multiplication, we can normalize a state (i.e. arrange that $\langle\Psi|\Psi\rangle = 1$). By further multiplying by a phase, we can arrange that the coefficient of $|\uparrow\rangle$ is real and positive. So we can represent any state as in (1), but with $|z_1|^2 + |z_2|^2 = 1$ and z_1 real an positive.

Problem 2 (Webwork)

For the state

$$|\Psi\rangle = i|\uparrow\rangle - \sqrt{3}|\downarrow\rangle \tag{2}$$

if we want to describe the same state using the form (1), but with $|z_1|^2 + |z_2|^2 = 1$ and z_1 real and positive, what are z_1 and z_2?

As we showed in class, the general state of our electron can always be written as

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle \tag{3}$$

where we have $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

Problem 3 (Webwork)

For the state in problem 2, what are θ and ϕ?

Problem 3 (hand in)

If we make a large number of measurements of the same observable on equivalent states, we will obtain a distribution of possible results governed by these probabilities. The average of these results, known as the *expectation value* of the observable is denoted by $\bar{\mathcal{O}}$ or $\langle\mathcal{O}\rangle$ and equal to

$$\bar{\mathcal{O}} = \sum_n p_n \lambda_n \tag{4}$$

where p_n is the probability of finding λ_n.

For an electron in a state

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle \ : \tag{5}$$

a) What is the expectation value of S_z?

b) What is the expectation value of S_x?

c) What is the expectation value of S_y?

If possible, simplify your answer to express it in terms of sins and cosines of θ rather than $\theta/2$. For these questions, use only the basic results that we have covered in class so far (i.e. you are not allowed to use Pauli matrices, spin operators, etc...)

Problem 4 (hand in)

We have seen that for a quantum system with a two-dimensional Hilbert space, we need two parameters θ and ϕ to represent the most general state. For a quantum system with a three-dimensional Hilbert space, what is the minimum number of parameters required. If the orthonormal basis elements are $|1\rangle$, $|2\rangle$ and $|3\rangle$ how can you write the most general state in a way that is analogous to (5)?

Problem 5: not really a problem

Do the reading for week 2 posted on the course website and think about the following reading question:

Question For a two dimensional real vector space, what are some examples of linear maps from the vector space to itself? Can you describe these both mathematically and in geometrical language? If you can, come to class with a couple of specific examples in mind.