Multipart quantum systems

It is often the case that a quantum mechanical system that we are interested in has various independent degrees of freedom i.e. separate parts that each have their own state. This is the case in a system with many particles, but also in systems where a single particle has both x motion and y motion, or x, y, z motion and spin.

To understand the quantum description of multipart systems, consider an example where we have two particles in one dimension. Classically, each would have a definite value for position. We can describe these by introducing variables x_1 and x_2 describing the locations of the two particles. But in the quantum system, states with definite values for x_1 and x_2 are very special states - they are eigenstates for the two position operators $\hat{X}_1$ and $\hat{X}_2$. If we label these states as $|x_1 x_2\rangle$, then the general state will be a linear combination of these. We need a function $\psi(x_1, x_2)$ to describe the coefficients in such a linear combination, so we see that the wavefunction for the two-part system is a function of two variables rather than just two functions of one variable.

The key point here is that the basis for the combined system is in one-to-one correspondence with pairs $(|x_1\rangle, |x_2\rangle)$ of basis elements, one from each part. Mathematically, a vector space with basis constructed in this way is known as the *tensor product* or direct product of the the two smaller vector spaces.[1].

In a case where both parts are represented by finite dimensional Hilbert spaces, if we have d_1 basis elements $|n\rangle$ for the first part and d_2 basis elements $|N\rangle$ for the second part, there are $d_1 d_2$ basis elements for the tensor product Hilbert space. These are labeled either as $|nN\rangle$ or $|n\rangle \otimes |N\rangle$, where the $\otimes$ in the latter notation indicates that there is a bilinear operation that takes one vector from the first Hilbert space and one vector from the second Hilbert space and gives us a vector in the larger Hilbert space. We can define this operation on any two vectors $|\psi\rangle = \sum_n c_n |n\rangle$ and $|\Psi\rangle = \sum c_N |N\rangle$ by

$$|\psi\rangle \otimes |\Psi\rangle = (\sum_n c_n |n\rangle) \otimes (\sum c_N |N\rangle) \equiv \sum_{nN} c_n C_N |n\rangle \otimes |N\rangle \ . \tag{1}$$

Physically, this tensor product state represents a state where the first subsystem is definitely in the state $|\psi\rangle$ while the second part is definitely in the state $|\Psi\rangle$. A very important point is that only very special states can be written in this way. For example, in a system with two spins, the state

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle) \tag{2}$$

cannot be written as a tensor product; in this case we say that the first subsystem is *entangled* with the second subsystem.

[1] This should be distinguished with a direct sum, in which the set of basis elements for the full space is taken to be the union of basis elements for the subspaces

We emphasize that the two vector spaces need not have the same dimension. For example, when describing a particle with position degrees of freedom and also spin degrees of freedom, we would have basis elements $|xs_z\rangle \equiv |x\rangle \otimes |s_z\rangle$ where the first part is infinite dimensional and the second part is finite dimensional. For a system like this, the general state would be a superposition

$$\int dx(\psi_\uparrow(x)|x\uparrow\rangle + \psi_\downarrow(x)|x\downarrow\rangle) \tag{3}$$

so we would have an "up" wavefunction and a "down" wavefunction to describe the state.

For a multipart system, given any operator acting on one of the individual Hilbert spaces, we can promote it to an operator acting on the full Hilbert space simply by declaring that it has no effect (i.e. that it acts as the identity operator) on the remaining parts. In our example with two finite-dimensional systems, if $\hat{\mathcal{O}}^1$ is an operator acting on the first part as

$$\hat{\mathcal{O}}^1|n\rangle = \sum_m \mathcal{O}^1_{mn}|m\rangle \ , \tag{4}$$

we can promote it to an operator acting on the whole system as

$$\hat{\mathcal{O}}^1|nN\rangle \equiv \sum_m \mathcal{O}^1_{mn}|mN\rangle \ . \tag{5}$$

Sometimes the full operator is denoted by $\hat{\mathcal{O}}_1 \otimes \mathbb{1}$ to emphasize that it is acting as the identity operator on the second part.

It is easy to check from the definition that operators defined in this way but acting on the different parts automatically commute with one another. Thus, we can write the product of operators acting on the different parts as $\mathcal{O}^1\mathcal{O}^2$ or as $\mathcal{O}^2\mathcal{O}^1$; in the $\otimes$ notation, both of these are equivalent to $\mathcal{O}^1 \otimes \mathcal{O}^2$. As an example, consider the operator $\hat{x}\hat{S}_x$ in our system describing the position and the spin of a particle. This is the product of the operator obtained by promoting $\hat{x}$ to an operator on the full system with the operator obtained by promoting $\hat{S}_x$ to an operator acting on the full system. Acting on an (x, S_z) basis state $|x\downarrow\rangle$, this gives

$$\hat{x}\hat{S}_x|x\downarrow\rangle = \frac{\hbar}{2}x|x\uparrow\rangle \ . \tag{6}$$

Just as the states of the combined system cannot usually be written in the form $|\Psi_1\rangle \otimes |\Psi_2\rangle$, general operators of the combined system cannot usually be written as $\mathcal{O}^1 \otimes \mathcal{O}^2$. Typically, the best we can do is write general states/operators as linear combinations of things that can be written in this way.