Translations and momentum

Another important example of an observable being related to some physical transformation is the connection between translations and momentum in quantum systems. Let $\hat{\mathcal{T}}(a)$ be the operator that translates a system by an amount a in some direction. Then we can write the infinitesimal version of this transformation as

$$\hat{\mathcal{T}}(a) = \mathbf{1} - ia\frac{1}{\hbar}\hat{P} + \dots \; . \tag{1}$$

By our general discussion above, the Hermitian operator $\hat{P}$ will correspond to a physical quantity that is conserved for any system where these translations are a symmetry. From classical mechanics, we know that this is the defining property of *momentum*, so we can say that $\hat{P}$ is the quantum operator associated with momentum. We have again included the constant $\hbar$ into the definition above in order that $\hat{P}$ will have the usual units of momentum.

For a system with a single spatial direction labeled by coordinate x, we can understand better the properties of the momentum operator by noting that if a state $|\Psi\rangle$ is defined by wavefunction $\psi(x)$ the the state $\hat{\mathcal{T}}(a)|\Psi\rangle$ must have wavefunction $\psi(x-a)$ i.e. the same function translated in the x direction by an amount a. This is implied by the definition of $\hat{\mathcal{T}}(a)$. Mathematically, this gives

$$\langle x|\hat{\mathcal{T}}(a)|\Psi\rangle = \psi(x-a) \; . \tag{2}$$

This equation is true for all a, so it must be true for very small a where (1) holds. in this case, we can expand both sides in a Taylor series in a, and all the terms must be equal. Inserting (1) into (2) and equating the terms with one power of a, we find

$$-ia\frac{1}{\hbar}\langle x|\hat{P}|\Psi\rangle = -a\psi'(x) \tag{3}$$

or

$$\langle x|\hat{P}|\Psi\rangle = -i\hbar\frac{d}{dx}\psi(x) \; . \tag{4}$$

We see that acting with the momentum operator on a state $|\Psi\rangle$ is equivalent to acting with $-id/dx$ on the wavefunction for that state.

Using this result, one application is to derive the commutator between the position operator and the momentum operator corresponding to that same direction. To do this, we note that the position operator $\hat{X}$ obeys $\hat{X}|x\rangle = x|x\rangle$, so for any $|\Psi\rangle$,

$$\langle x|\hat{X}\hat{P}|\Psi\rangle = x\langle x|\hat{P}|\Psi\rangle = -i\hbar x\frac{d}{dx}\psi(x) \; . \tag{5}$$

On the other hand, the wavefunction for the state $\hat{X}|\Psi\rangle$ is

$$\langle x|\hat{X}|\Psi\rangle = x\langle x|\Psi\rangle = x\psi(x) \tag{6}$$

so by our result above, the wavefunction $\langle x|\hat{P}\hat{X}|\Psi\rangle$ for the state $\hat{P}(\hat{X}|\Psi\rangle)$ is

$$-i\frac{d}{dx}(x\psi(x)) = -i\hbar\psi(x) - i\hbar x\frac{d}{dx}\psi(x) \ . \tag{7}$$

Thus, we finally have that

$$\langle x|[\hat{X},\hat{P}]|\Psi\rangle = i\hbar\psi(x) = i\hbar\langle x|\Psi\rangle \ . \tag{8}$$

This implies that the matrix elements of the operator $[\hat{X},\hat{P}]$ are the same as the matrix elements of the operator $i\hbar\mathbf{1}$, so it must be that

$$[\hat{X},\hat{P}] = i\hbar\mathbf{1} \ . \tag{9}$$

This fundamental relation will be very useful in understanding a variety of quantum mechanical systems. Based on the generalized uncertainty principle above, this tells us immediately that

$$\Delta X\Delta P \geq \frac{\hbar}{2} \ . \tag{10}$$

Momentum eigenstates

As an application of our results, we can understand the properties of particle states that have a definite momentum. Suppose we have an eigenstate $|p\rangle$ satisfying

$$\hat{P}|p\rangle = p|p\rangle \ . \tag{11}$$

We can turn this into an equation for the particle's position wavefunction by taking the inner product with a general position basis state $|x\rangle$. We get

$$\begin{aligned} & \langle x|\hat{P}|p\rangle = \langle x|p|p\rangle \\ \implies \quad & \frac{\hbar}{i}\frac{d\psi_p}{dx} = p\psi_p(x) \\ \implies \quad & \psi_p(x) = Ce^{\frac{ipx}{\hbar}} \end{aligned}$$

So we see that the wavefunction for a particle with momentum p is sinusoidal with wavelength h/p, reproducing the famous result of de Broglie. These momentum eigenstates are not normalizible; this is related to the constraint from the uncertainty principle that the position uncertainty must be infinite if we have zero momentum uncertainty.

To choose the constant C, we normally require instead that the states satisfy

$$\langle p|q\rangle = \delta(p-q) \ . \tag{12}$$

This gives

$$\int_{-\infty}^{\infty} dx C^2 e^{\frac{-ipx}{\hbar}} e^{\frac{iqx}{\hbar}} = \delta(p-q)$$

$$\implies \quad C^2\hbar\delta(p-q) = \delta(p-q)$$

so we can normalize the momentum eigenstate wavefunctions as

$$\psi_p(x) = \frac{1}{\sqrt{\hbar}} e^{\frac{ipx}{\hbar}} \ . \tag{13}$$

Using this normalization, we can check that $|\langle p|\psi\rangle|^2$ will always integrate to 1, so

$$\chi(p) \equiv \langle p|\psi\rangle = \int_{-\infty}^{\infty} \frac{dx}{\hbar} e^{\frac{-ipx}{\hbar}} \psi(x) \tag{14}$$

has the interpretation of a momentum space wavefunction whose square gives the probability density for measuring various values of momentum.

Example: a particle in one dimension

As an example for how to use all of this formalism, consider a physical system describing a particle in one dimension subject to forces corresponding to a potential energy function $V(x)$. In classical mechanics, the energy for such a system is given by

$$E = \frac{p^2}{2m} + V(x) \ . \tag{15}$$

For the quantum system, time evolution is governed by the Hamiltonian, which is the same as the energy operator. In quantum mechanics, position and momentum can't have definite values at the same time, so at best, the relation (15) could be true about the expectation values of these quantities. We can arrange this by translating (15) to a statement about operators

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{X}) \ . \tag{16}$$

With this definition, we can now translate the general form (**??**) for the Schrödinger equation to a differential equation describing the evolution of the position-space wavefunction $\psi(x)$. We'll call the wavefunction at time t $\psi(x,t)$. Starting from (**??**), we can take the inner product on both sides with the position space basis element $|x\rangle$ to obtain

$$\begin{aligned}
& \langle x|i\hbar\frac{d}{dt}|\Psi\rangle = \langle x|\hat{H}|\Psi\rangle \\
\iff \quad & i\hbar\frac{d}{dt}\langle x|\Psi\rangle = \langle x|\frac{\hat{P}^2}{2m} + V(\hat{X})|\Psi\rangle \\
\iff \quad & i\hbar\frac{\partial}{\partial t}\psi(x,t) = \frac{1}{2m}\langle x|\hat{P}^2|\Psi\rangle + \langle x|V(\hat{X})|\Psi\rangle \\
\iff \quad & i\hbar\frac{\partial}{\partial t}\psi(x,t) = \frac{1}{2m}(-i\hbar\frac{d}{dx})^2\langle x|\Psi\rangle + \langle x|V(x)|\Psi\rangle
\end{aligned}$$

$$\iff \quad i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t) \tag{17}$$

Thus, we have derived the usual position-space form of the 1D Schrödinger equation, appearing on page 1 of Griffiths.