Solving the Schrödinger Equation

A useful consequence of this is that to understand time-evolution in quantum mechanics, it is very useful to find the eigenstates of the energy operator. If $|E_n\rangle$ represent these energy eigenstates, we have by definition

$$\hat{H}|E_n\rangle = E_n|E_n\rangle \tag{1}$$

so the time evolution operator acts as[1]

$$\hat{\mathcal{T}}(t)|E_n\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|E_n\rangle = e^{-\frac{i}{\hbar}E_n t}|E_n\rangle \tag{2}$$

We see that the state of the system after time t is physically equivalent to the original state, since it is just the original state multiplied by a phase factor. In particular, the probabilities for any possible measurements will be the same as for the initial state. For this reason, we call the energy eigenstates *stationary states*.

Using the linearity property of time evolution in quantum mechanics, it is now simple to understand the evolution of any state. We first take the state at $t = 0$ and express it as a combination of energy eigenstates:

$$|\Psi(t=0)\rangle = \sum c_n|E_n\rangle \qquad\qquad c_n = \langle E_n|\Psi(t=0)\rangle\ . \tag{3}$$

By linearity, we can immediately say that the state at a later time is

$$|\Psi(t)\rangle = \hat{\mathcal{T}}(t)|\Psi(t=0)\rangle = \sum c_n\hat{\mathcal{T}}(t)|E_n\rangle = \sum c_n e^{-\frac{i}{\hbar}E_n t}|E_n\rangle\ . \tag{4}$$

Thus, the problem of understanding time evolution is reduced to the problem of finding the energy eigenstates, i.e. solving (1). For this reason, (1) is often called the time-independent Schrödinger equation.

Translations and momentum

Another important example of an observable being related to some physical transformation is the connection between translations and momentum in quantum systems. Let $\hat{\mathcal{T}}(a)$ be the operator that translates a system by an amount a in some direction. Then we can write the infinitesimal version of this transformation as

$$\hat{\mathcal{T}}(a) = \mathbf{1} - ia\frac{1}{\hbar}\hat{P} + \dots\ . \tag{5}$$

By our general discussion above, the Hermitian operator $\hat{P}$ will correspond to a physical quantity that is conserved for any system where these translations are a symmetry. From

[1]We are using here that if a state is an eigenstate of some operator, it will be an eigenstate of any power of that operator and more generally any (analytic) function of that operator. Alternatively, we can just check that our final result satisfies the Schrödinger equation.

classical mechanics, we know that this is the defining property of *momentum*, so we can say that $\hat{P}$ is the quantum operator associated with momentum. We have again included the constant $\hbar$ into the definition above in order that $\hat{P}$ will have the usual units of momentum.

For a system with a single spatial direction labeled by coordinate x, we can understand better the properties of the momentum operator by noting that if a state $|\Psi\rangle$ is defined by wavefunction $\psi(x)$ the the state $\hat{\mathcal{T}}(a)|\Psi\rangle$ must have wavefunction $\psi(x-a)$ i.e. the same function translated in the x direction by an amount a. This is implied by the definition of $\hat{\mathcal{T}}(a)$. Mathematically, this gives

$$\langle x|\hat{\mathcal{T}}(a)|\Psi\rangle = \psi(x-a) \; . \tag{6}$$

This equation is true for all a, so it must be true for very small a where (5) holds. in this case, we can expand both sides in a Taylor series in a, and all the terms must be equal. Inserting (5) into (6) and equating the terms with one power of a, we find

$$-ia\frac{1}{\hbar}\langle x|\hat{P}|\Psi\rangle = -a\psi'(x) \tag{7}$$

or

$$\langle x|\hat{P}|\Psi\rangle = -i\hbar\frac{d}{dx}\psi(x) \; . \tag{8}$$

We see that acting with the momentum operator on a state $|\Psi\rangle$ is equivalent to acting with $-id/dx$ on the wavefunction for that state.