Expectation values and Uncertainty

If we make a large number of measurements of the same observable on equivalent states, we will obtain a distribution of possible results governed by these probabilities. The average of these results, known as the *expectation value* of the observable is denoted by $\bar{\mathcal{O}}$ or $\langle \mathcal{O} \rangle$ and equal to

$$\bar{\mathcal{O}} = \sum_n p_n \lambda_n = \sum_n \lambda_n |\langle \lambda_n | \Psi \rangle|^2 \tag{1}$$

We may also be interested in how spread out the distribution of results is. A useful measure of this is to compute the standard deviation of the distribution, defined by taking the average value of $(\mathcal{O} - \bar{\mathcal{O}})^2$ and then taking the square root. In quantum mechanics, we call this the *uncertainty* $\Delta\mathcal{O}$ of the observable $\mathcal{O}$ in the state $|\Psi\rangle$. It is given explicitly by

$$\Delta\mathcal{O} = (\sum_n p_n (\lambda_n - \bar{\mathcal{O}})^2)^{\frac{1}{2}} \tag{2}$$

where we first calculate $\bar{\mathcal{O}})$ using (1).

Hermitian operators for observables

Given any observable $\mathcal{O}$, we can define a Hermitian operator $\hat{\mathcal{O}}$ associated with it, by defining

$$\hat{\mathcal{O}}|\lambda_n\rangle = \lambda_n |\lambda_n\rangle \tag{3}$$

for the eigenstates of $\mathcal{O}$ and using linearity to define the action on any other state. By our definition, the eigenvectors of $\hat{\mathcal{O}}$ are orthonormal and the eigenvalues are real, so the operator is Hermitian.

We will see below what the physical interpretation of the state $\hat{\mathcal{O}}|\Psi\rangle$ is; for now we point out the useful result that the expectation value of $\mathcal{O}$ defined above can be calculated for the state Ψ as

$$\bar{\mathcal{O}} = \langle \Psi | \hat{\mathcal{O}} | \Psi \rangle \ . \tag{4}$$

This can be checked using the definition (3) to reproduce the result (3).

Commuting vs noncommuting observables

For two different observables $\mathcal{O}_1$ and $\mathcal{O}_2$, it is typically the case that the eigenvectors of one are not eigenvectors of the other. Physically, this means that a state with a definite value for $\mathcal{O}_1$ will not have a definite value for $\mathcal{O}_2$ and vice versa.

On the other hand, if the two observables do share a common basis of eigenstates, it is possible to know the value of both observables at the same time.

It is not hard to show that two Hermitian operators will share a common basis of eigenstates if and only if the corresponding operators *commute* with each other, i.e. that in acting with the two operators in succession on any state, we get the same result regardless of the order. This will be true if and only if the *commutator* $[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2] \equiv \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 - \hat{\mathcal{O}}_2, \hat{\mathcal{O}}_1$ is equal to zero.

In the case where the operators share a common basis, both $\mathcal{O}_1$ and $\mathcal{O}_2$ will have matrix representations that are diagonal in this basis, so it is clear that the corresponding matrices also commute with each other.

The fact that some observables cannot have definite values simultaneously leads directly to the idea of an *uncertainty principle* in quantum mechanics. Starting from the definition (2) of uncertainty and making use of the Cauchy-Schwarz inequality (see Griffiths for details) it is possible to prove the *generalized uncertainty principle*

$$\Delta \mathcal{O}_1 \Delta \mathcal{O}_2 \geq \frac{1}{2} |\langle i[\mathcal{O}_1, \mathcal{O}_2] \rangle| \; . \tag{5}$$

We will see that this leads to the more familiar Heisenberg Uncertainty Principle when applied to the position and momentum operators.