## The Qubit

The simplest non-trivial quantum system is known as a **qubit**. A qubit refers to any quantum system whose states are vectors in a two-dimensional Hilbert space. An important example is the physical system describing the spin states of an electron. To define this, we can imagine that an electron is in some particular fixed position state (e.g. in the ground state of a Coulomb potential) that doesn't change (e.g. because there isn't enough energy around to excite it to some other state). Even with a fixed position, there are still a family of states available to the electron, which correspond to the possible orientations of its spin. It turns out that these possible states form a two-dimensional Hilbert space.

It is helpful to work with a specific basis for this vector space. We talked in class about how for any physical observable, there is a basis of states that have definite values for that observable. Since spin refers to the intrinsic angular momentum of a particle, the most natural observable to consider in order to distinguish different spin states is the angular momentum about some axis. For starters, we'll consider $S_z$, the electron's spin angular momentum about the $z$ axis.

According to our basic assumptions, we can find an orthonormal basis of states such that the basis elements have definite values for $S_z$. In the past, you have learned that these possible values are $\hbar/2$ and $-\hbar/2$. We'll call the basis state with $S_z = \hbar/2$ $|\uparrow\rangle$ and the basis state with $S_z = -\hbar/2$ $|\downarrow\rangle$. So we have

$$\langle\uparrow|\uparrow\rangle = 1 \qquad \langle\uparrow|\downarrow\rangle = 0 \qquad \langle\downarrow|\uparrow\rangle = 0 \qquad \langle\downarrow|\downarrow\rangle = 1$$

A general state for the electron's spin can then be written as

$$|\Psi\rangle = z_1|\uparrow\rangle + z_2|\downarrow\rangle \; . \tag{1}$$

We said that vectors related through multiplication by a nonzero complex number represent the same state. So a state (1) defined by coefficients $(z_1, z_2)$ is the same as a state defined by coefficients $(wz_1, wz_2)$. By such a multiplication, we can normalize a state (i.e. arrange that $\langle\Psi|\Psi\rangle = 1$). By further multiplying by a phase, we can arrange that the coefficient of $|\uparrow\rangle$ is real and positive. So we can represent any state as in (1), but with $|z_1|^2 + |z_2|^2 = 1$ and $z_1$ real an positive.

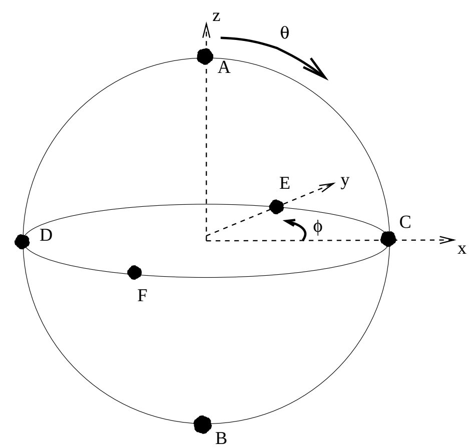
Any positive numbers that sum to one can be represented as $\cos^2(\theta/2)$ and $\sin^2(\theta/2)$ for some $\theta$ in $[0, \pi]$.[1] So for $(z_1, z_2)$ satisfying $|z_1|^2 + |z_2|^2 = 1$, we can pick $\theta$ so that $|z_1| = \cos(\theta/2)$ and $|z_2| = \sin(\theta/2)$. Assuming that $z_1$ is real and positive, we must have $z_1 = \cos(\theta/2)$, while for $z_2$, we could have generally $z_2 = e^{i\phi}\sin(\theta/2)$. So the general state of our electron can always be written as

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle \tag{2}$$

[1] We could have just used $\theta$ instead of $\theta/2$ inside the trigonometric functions, but it will be nicer to have $\theta$ taking the range $[0, \pi]$ for reasons that we'll see below.

where we have $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

A nice thing about this way of representing states is that the parameters and parameter ranges are exactly the ones we would use to describe points on a sphere. We see that the states $|\uparrow\rangle$ and $|\downarrow\rangle$ correspond to $\theta = 0$ and $\theta = \pi$, so these are the north pole and south pole of the sphere. It turns out that the other points on the sphere correspond to eigenstates of other spin angular momentum operators pointing in different directions.

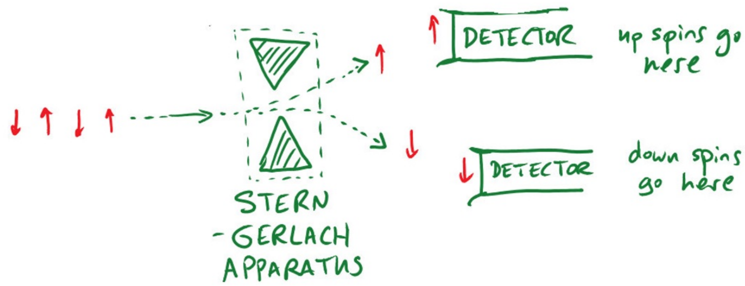



The sphere in the figure represents the space of distinct quantum states of a qubit. For our example of the electron spin the states $|\uparrow\rangle$ and $|\downarrow\rangle$ with $S_z = \pm\hbar/2$ correspond to the north and south poles (points A and B). The four states below correspond to the remaining points C,D,E, and F on the sphere, respectively.

$$
\begin{aligned}
|\Psi_1\rangle &= \frac{\sqrt{2}}{2}(|\uparrow\rangle + |\downarrow\rangle) \\
|\Psi_2\rangle &= \frac{\sqrt{2}}{2}(|\uparrow\rangle - |\downarrow\rangle) \\
|\Psi_3\rangle &= \frac{\sqrt{2}}{2}(|\uparrow\rangle + i|\downarrow\rangle) \\
|\Psi_4\rangle &= \frac{\sqrt{2}}{2}(|\uparrow\rangle - i|\downarrow\rangle)
\end{aligned}
$$

Physically, the states corresponding to points $C$ and $D$ are eigenstates of $S_x$ with eigenvalues $\pm\hbar/2$, while the states corresponding to points $E$ and $F$ are eigenstates of $S_y$ with eigenvalues $\pm\hbar/2$. More generally, a state corresponding to any point on the sphere will be an eigenstate for a physical observable which is the component of the angular momentum (spin) in the direction corresponding to that point on the sphere. So the sphere represents the direction of the spin in physical 3D space.

Apart from $|\uparrow\rangle$ and $|\downarrow\rangle$, states of the electron don't have a definite value for $S_z$. Let's recall the physical implications of this. Suppose we set up an experiment where we send a beam of electrons through a configuration of magnets such that electrons in spin state $|\uparrow\rangle$ end up in one detector (detector A) while electrons in spin state $|\downarrow\rangle$ end up in another detector




(detector B). What would happen if we now send in a beam of electrons whose spin state is the one with $S_x = \hbar/2$? Since this state $|\rightarrow\rangle$ can be written as

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle \ , \tag{3}$$

and since the experiment constitutes a measurement of the $z$ componenet of the spin, we can say that the particiles will either be measured in the up detector or the down detector (i.e. $z$ spin will be measured as up or down) with probabilities given by the squared magnitudes of the coefficients in the superposition, i.e. with probability 1/2 in this case.

**Many qubits**

What if we want to describe the state of many qubits (e.g. the spin configuration of a collection of spin 1/2 particles at fixed positions). In this case, one basis for the Hilbert space will be the set of states where each spin has a specific value for $S_z$. Since there are two choices for each spin, we have $2^N$ different basis elements. The general state is a linear combination of these basis elements, so we need $2^N$ complex numbers $z_i$ to describe such a state. Even after fixing the normalization to be 1 and multiplying by a phase (e.g. to make the first coefficient real), we still need $2 \cdot 2^N - 2$ real parameters to desccribe the most general state. So for example, the different states of 100 qubits form a $(2^{101} - 2) =$ 2535301200456458802993406410750 dimensional space!