1 Basics of Quantum mechanics

Quantum mechanics is a mathematical framework that is believed to underlie all physics in our universe. In these notes, we review the fundamentals of this framework. Each time a new mathematical concept is used, we provide a review of that concept in an indented section just before we need it. Feel free to skip ahead to the physics parts and go back and read all the definitions when you need them. We begin with the definition of a Hilbert space.

Hilbert space

Complex vector spaces

We start with the idea of a *complex vector space*. This is a space of objects that we call *vectors*, such that any complex number times a vector gives another vector and any sum of vectors gives another vector. In quantum mechanics, we will use the notation $|v\rangle$ (called a "ket") to denote a vector. All the properties familiar from real vector spaces still hold in the complex case.[1]

Inner product

For the vector spaces in quantum mechanics, we have an additional structure called an *inner product* which generalizes the dot product for real vector spaces. The inner product is a map that takes a pair of vectors to a complex number that we denote by

$$(|v_1\rangle, |v_2\rangle) \to \langle v_1|v_2\rangle \tag{1}$$

The inner product obeys the following properties:

- $\langle v_2|v_1\rangle = \langle v_1|v_2\rangle^*$
- If $|w\rangle = z_1|v_1\rangle + z_2|v_2\rangle$ then $\langle v_3|w\rangle = z_1\langle v_3|v_1\rangle + z_2\langle v_3|v_2\rangle$
- If $|w\rangle = z_1|v_1\rangle + z_2|v_2\rangle$ then $\langle w|v_3\rangle = z_1^*\langle v_1|v_3\rangle + z_2^*\langle v_2|v_3\rangle$
- $\langle v|v\rangle \geq 0$ with equality only for $|v\rangle = \mathbf{0}$.

Here, the third property follows from the first two. The last property allows us to associated a real "length" to a vector and also gives us a measure of distance between vectors, defined as the length of the difference vector.

[1] Specifically, we have $(|v_1\rangle + |v_2\rangle) + |v_3\rangle = |v_1\rangle + (|v_2\rangle + |v_3\rangle)$, $|v_1\rangle + |v_2\rangle = |v_2\rangle + |v_1\rangle$, there exists $\mathbf{0}$ such that $|v\rangle + \mathbf{0} = |v\rangle$ for all $|v\rangle$, $-|v\rangle + |v\rangle = \mathbf{0}$, $z_1(z_2|v\rangle) = (z_1 z_2)|v\rangle$, $1|v\rangle = |v\rangle$, $z(|v_1\rangle + |v_2\rangle) = z|v_1\rangle + z|v_2\rangle$, $(z_1 + z_2)|v\rangle = z_1|v\rangle + z_2|v\rangle$.

Dual vectors

We can also give meaning to the object $\langle v|$, (known as a "bra"). It represents a linear map from the vector space to complex numbers, known mathematically as a *dual vector*. The map is defined by

$$|w\rangle \to \langle v|w\rangle \ . \tag{2}$$

Any linear map from our vector space to complex numbers can be represented in this way for some choice of state $|w\rangle$.[2]

Hilbert space

Finally we can define a complex *Hilbert space*: it is a complex vector space with inner product satisfying one more technical criterion known as *completeness*. This is the statement that any sequence of vectors whose elements become arbitrarily close to one another as the sequence progresses (a *Cauchy sequence*) must converge to a vector in the space; this is automatically satisfied for finite dimensional examples.

1.1 States

The physical configuration of a system at some time is known as a *state*. In quantum mechanics, a state is represented mathematically as a nonzero **vector** in some **Hilbert space** . We will often denote this as $|\Psi\rangle$. Two vectors represent the same physical state if they are related to each other through multiplication by a complex number.[3] Since the zero vector is not allowed, any state vector is equivalent to a state with $\langle\Psi|\Psi\rangle = 1$, so we will often assume this condition and refer to such states as *normalized*. There is still freedom to multiply by a complex number $e^{i\phi}$ with norm 1 (known as a *phase*) without changing the physical configurations.

Generally, this state will evolve with time, so we can write $|\Psi(t)\rangle$ to describe the state vector at time t.

Orthonormal bases, finite and infinite dimensional spaces

It is often convenient (and will be physically relevant) to define a basis for a vector space. We define an *orthonormal basis* to be a set of vectors $|e_n\rangle$ which

[2]Proof: for an orthonormal basis $|e_i\rangle$, let z_i be the result of applying the map to $|e_i\rangle$. The map on any other state can be determined from these numbers using the linearity relation. We can now check that the map is equivalent to taking the inner product with the state $\sum_i z_i^* |e_i\rangle$.

[3]Mathematically, the set of nonzero vectors in an N-dimensional complex vector space $\mathbb{C}^N$ together with this equivalence relation defines a space called complex projective space $\mathbb{CP}^{N-1}$.

span the space (i.e. any vector can be expressed as a linear combination of them) and such that

$$\langle e_m|e_n\rangle = \delta_{mn} \ . \tag{3}$$

Here δ_{mn} vanishes for m not equal to n and is 1 otherwise. Given any vector $|v\rangle$, we can then write

$$|v\rangle = \sum_n c_n|e_n\rangle \tag{4}$$

where

$$c_n = \langle e_n|v\rangle \ . \tag{5}$$

We say that the set of coefficients $(c_1, c_2, c_3, \dots)$. represent the vector $|v\rangle$ in the basis $|e_n\rangle$. There are an infinite number of possible bases, and the set of coefficients representing a given vector will generally be different for each choice. Thus, it is important to keep in mind that the list $(c_1, c_2, c_3, \dots)$ has no intrinsic meaning unless we specify which basis we are talking about.

Dimension of Hilbert space

The number of basis elements can be finite or infinite. When the number is finite, every orthonormal basis for the vector space will have the same number of vectors. We call this the *dimension* of the Hilbert space.

1.2 Observables

In quantum mechanics, physical quantities such as energy, position, or angular momentum (we will generally call them *observables*) do not generally have definite values in a given state. However, a basic assumption is that there exists for each observable $\mathcal{O}$ some orthonormal basis of states $|\lambda_n\rangle$, each of which has a definite value λ_n for $\mathcal{O}$. We call these the eigenvectors (or eigenstates) and eigenvalues associated with $\mathcal{O}$. In some cases, two or more of the eigenvalues can be the same; in this case, the whole space of states spanned by the corresponding eigenvectors is understood to have this same definite value for $\mathcal{O}$, and we have some freedom in choosing which orthonormal basis of states to choose.

What about states which are not eigenstates? Since the eigenstates form a basis, we can write any state $|\Psi\rangle$ as a linear combination of these basis elements,

$$|\Psi\rangle = \sum_n c_n|\lambda_n\rangle \ . \tag{6}$$

For such a state, we say that $|\Psi\rangle$ does not have a definite value for $\mathcal{O}$; in a measurement of $\mathcal{O}$, we might find any of the values λ_n, with probability $|c_n|^2$, and the state becomes the corresponding eigenstate after the measurement. Here, we are assuming that the state is normalized; otherwise $|c_n|^2$ gives the relative probability.

To see that the probabilities add up to 1 in the normalized case, we note that by the orthonormality of the basis,

$$1 = \langle\Psi|\Psi\rangle = \sum_n \sum_m c_n^* c_m \langle\lambda_n|\lambda_m\rangle = \sum_n |c_n|^2 = 1 \ . \tag{7}$$

We can give a more direct formula for the probability by noting that

$$c_n = \langle\lambda_n|\Psi\rangle \ . \tag{8}$$

which follows using the orthonormality of the basis by taking the inner product of $|\lambda_n\rangle$ with the two sides of (6). Thus if we measure $\mathcal{O}$ in the state $|\Psi\rangle$, the probability that we will obtain the result λ_n is simply

$$p_n = |\langle\lambda_n|\Psi\rangle|^2 \ . \tag{9}$$