The Harmonic Oscillator, a Review

Here, we review the physics of the one-dimensional harmonic oscillator, a quantum system describing a 1D particle with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \ . \tag{1}$$

As we have seen, a key problem is to understand the energy eigenstates of this Hamiltonian, i.e. the states satisfying

$$\hat{H}|\Psi\rangle = E|\Psi\rangle \tag{2}$$

These will allow us to give the general solution of the Schrödinger equation for the time evolution of states.

We can actually accomplish this without ever using wavefunctions or the position basis if we simply make use of the basic relation

$$[\hat{x}, \hat{p}] = i\hbar \ . \tag{3}$$

The trick is to define the *annihilation operator*

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\hat{p} \tag{4}$$

and its adjoint, the *creation operator*

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\hat{p} \ . \tag{5}$$

Then using (3) and these definitions, we find that[1]

$$[\hat{a}, \hat{a}^\dagger] = \mathbf{1} \tag{6}$$

and

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \ . \tag{7}$$

Making use of these results, we can show that

$$[\hat{H}, \hat{a}] = -\hbar\omega\hat{a} \qquad\qquad [\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger \tag{8}$$

Now, suppose that $|E\rangle$ is an energy eigenstate. We will now show that $\hat{a}|E\rangle$ is an energy eigenstate with energy $E - \hbar\omega$ and $\hat{a}^\dagger|E\rangle$ is an energy eigenstate with energy $E + \hbar\omega$. To see this, we use (8) to obtain

$$\hat{H}(\hat{a}|E\rangle) = ([\hat{H}, \hat{a}] + \hat{a}\hat{H})|E\rangle = -\hbar\omega\hat{a}|E\rangle + \hat{a}E|E\rangle = (E - \hbar\omega)\hat{a}|E\rangle \ . \tag{9}$$

[1] We recall also that the commutator of any operator with itself is zero by the definition of the commutator, so $[a, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$.

A similar calculation shows that $\hat{H}(\hat{a}^\dagger|E\rangle) = (E + \hbar\omega)\hat{a}^\dagger|E\rangle$. This may be a little disconcerting, since it looks like we could get states with an arbitrarily large negative energy by acting with $\hat{a}$ enough times.

But this can't be right: if $|E\rangle$ is an energy eigenstate, then

$$\begin{aligned} E &= \langle E|\hat{H}|E\rangle \\ &= \frac{1}{2}\hbar\omega + \langle E|\hat{a}^\dagger\hat{a}|E\rangle \\ &\geq \frac{1}{2}\hbar\omega \end{aligned} \tag{10}$$

where we have used the fact that $\langle E|\hat{a}^\dagger\hat{a}|E\rangle$ is the inner product of $\hat{a}|E\rangle$ with itself, which must be nonnegative and vanish only if $\hat{a}|E\rangle = 0$. So all the energy eigenvalues must be positive.

The only way to avoid our earlier conclusion about having arbitrarily negative energies would be if by acting with $\hat{a}$s on any energy eigenstate we eventually get to a state where $\hat{a}|0\rangle$ acting on this state gives 0. By (10), the energy of such a state is equal to $\hbar\omega/2$. As we'll show below, there is a unique state with this property, and the normalized state is usually called $|0\rangle$, not to be confused with the zero vector.

Starting from $|0\rangle$, we can generate all the other energy eigenstates by acting with the creation operator. We define

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle \tag{11}$$

where the constant is chosen so that the states are properly normalized. Our earlier calculations then show that these have energies

$$E_n = \hbar\omega(n + \frac{1}{2}) \ . \tag{12}$$

With these properly normalized states, we find that

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \qquad \hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle \ . \tag{13}$$

To complete the discussion, we need to show that $\hat{a}|0\rangle = 0$ specifies the state uniquely. We will do this by showing that it completely determines the position space wavefunction $\psi_0(x) = \langle x|\psi\rangle$ for the state. Using $\hat{a}|0\rangle = 0$, we have

$$\begin{aligned} & \langle x|\hat{a}|0\rangle = 0 \\ \Longrightarrow \quad & \langle x|\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}|0\rangle = 0 \\ \Longrightarrow \quad & \sqrt{\frac{m\omega}{2\hbar}}\langle x|\hat{x}|0\rangle + \frac{i}{\sqrt{2m\omega\hbar}}\langle x|\hat{p}|0\rangle = 0 \\ \Longrightarrow \quad & \sqrt{\frac{m\omega}{2\hbar}}x\psi_0(x) + \frac{i}{\sqrt{2m\omega\hbar}}\frac{\hbar}{i}\frac{d}{dx}\psi_0(x) = 0 \end{aligned} \tag{14}$$

Solving this equation, we have that

$$\psi_0(x) = \left(\frac{\pi\hbar}{m\omega}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \; . \tag{15}$$

where the overall constant has been fixed by demanding a normalized wavefunction.

In performing calculations for harmonic oscillator states, we can often avoid using the position space wavefunctions by rewriting $\hat{x}$ and $\hat{p}$ operators in terms of creation and annihilation operators

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \qquad \hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a} - \hat{a}^\dagger) \tag{16}$$

and then using (6) and (13). For example,

$$\langle 2|\hat{x}|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 2|\hat{a} + \hat{a}^\dagger|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 2|(|0\rangle + \sqrt{2}|2\rangle) = \sqrt{\frac{\hbar}{m\omega}} \tag{17}$$