Physics 402 Handout: Charge Particles in Electromagnetic Fields

We would like to understand the Hamiltonian governing the behaviour of charged particles in general electric and magnetic fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$.

Classical Equations of Motion

To start, recall that clasically, the physics is governed by the Lorentz force law:

$$m\frac{d^2\vec{x}}{dt^2} = q(\vec{E}(\vec{x}(t),t) + \dot{\vec{x}}(t) \times \vec{B}(\vec{x}(t),t))$$

To find the Hamiltonian for this system, a first step is to find a Lagrangian that reproduces this equation of motion via the principle of least action. It turns out that it is impossible to write such a Lagrangian in terms of E and B. Rather, we must use the electromagnetic potential fields from which E and B can be derived.

Electromagnetic potentials and gauge invariance

Recall that since B has no divergence, we can always write it as

$$B = \nabla \times A$$

for some vector potential A. Also, since

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

or in other words

$$\nabla \times [E + \frac{\partial A}{\partial t}] = 0\ ,$$

the combination in brackets must be the gradient of some potential $-\phi$, so

$$E = -\nabla\phi - \frac{\partial A}{\partial t}\ .$$

The description of electromagnetic fields in terms of potentials is redundant. It is easy to check that if we have any potentials ϕ and A describing fields E and B then for any arbitrary function λ, the new potentials

$$\tilde{\phi} = \phi - \partial_t\lambda$$
$$\tilde{A} = A + \nabla\lambda$$

give the same electric and magnetic fields. These transformations, which change the potentials but do not change the fields, are called *gauge transformations.* Invariance under these gauge transformations is a fundamental property of electromagnetism, and is directly linked to the law that charge must be conserved.

Sometimes it is useful to use this freedom of making gauge transformations to impose an additional restriction on the potentials, for example to demand that

$$\nabla \cdot A = 0$$

This type of condition is known as a choice of gauge, and this particular choice is known as *Coulomb Gauge.*

Lagrangian for charged particles in electromagnetic fields

We are now ready to write the Lagrangian for a charged particle in general electromagnetic fields. For now, we will just write it down, and then check that it is correct. We take

$$L(x(t), \dot{x}(t)) = \frac{1}{2} m \dot{x}^2(t) - q\phi(x(t), t) + q\dot{x} \cdot A(x(t), t)$$

For this lagrangian, the principle of least action states that among all possible trajectories for a particle from x_i at time t_i to x_f at time t_f, the physical one is the one that minimizes the action

$$S = \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \, .$$

To find the conditions for minimizing the action more explicitly, we suppose that $x(t)$ is the minimum action trajectory, and consider the variation $x(t) + \delta(t)$. Plugging this into the action, we must demand that all terms linear in $\delta(t)$ cancel (this is like saying that $f'(x) = 0$ if x is the minimum of a function). In general, this implies that

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i} \, ,$$

which are known as the Euler-Lagrange equations. We can now check that for our specific choice of Lagrangian, these conditions that the action is minimized give exactly the equations of motion above.

Hamiltonian for charged particles

It is now straightforward to derive the Hamiltonian. According to the usual rules of classical mechanics. To get the Hamiltonian from the Lagrangian, we define

$$p_i = \frac{\partial L}{\partial \dot{x}^i}$$

and then take

$$H(p, x) = p \cdot \dot{x} - L \, .$$

In our case, we find that

$$p_i = m \cdot x_i + qA_i(x)$$

and using this to solve for $\dot{x}$ in terms of x and p, and plugging everything in to the formula for the Hamiltonian, we find

$$H(x, p) = \frac{1}{2m}(p - qA(x))^2 + q\phi(x)$$

This is the classical Hamiltonian. To go to quantum mechanics, we simply promote x_i and p_i to operators, satisfying the commutation relations

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar\delta_{ij}$$

Expanding the terms, we have finally that in the presence of electric and magnetic fields, we must add to the Hamiltonian of each charged particle the terms

$$H' = -\frac{q}{m}\mathbf{p} \cdot A(\mathbf{x}, t) + \frac{q^2}{2m} A^2(\mathbf{x}, t) + q\phi(\mathbf{x}, t)$$

Since the first term here includes both position and momentum operators, we should be concerned about their ordering, since generally

$$\mathbf{p} \cdot A(\mathbf{x}, t) \neq A(\mathbf{x}, t) \cdot \mathbf{p} \, .$$

However, the difference between these is

$$-i\hbar\nabla \cdot A(\mathbf{x}, t)$$

i.e. $\vec{\nabla} \cdot \vec{A} = 0$

so if we choose potentials to satisfy the Coulomb gauge condition, there are no problems.