Sample degenerate perturbation theory problem:

We have a system of 2 spins, with Hamitonian

$$H_0 = A\cdot\left(S_z^{(1)} + S_z^{(2)}\right)$$

We add a perturbation $H \to H + \lambda S_x^{(1)}\cdot S_x^{(2)}$ and want to find the eigenstate energies to order λ.

<u>SOLUTION</u>: For H_0, the eigenstates are

$$|\uparrow\uparrow\rangle : \quad E = A\cdot\hbar$$
$$|\uparrow\downarrow\rangle : \quad E = 0$$
$$|\downarrow\uparrow\rangle : \quad E = 0$$
$$|\downarrow\downarrow\rangle : \quad E = -A\cdot\hbar$$

Here $S_z^{(1)}$ is the z-component of the first spin, and $S_z^{(2)}$ is the z component of the second spin, so the states where both spins have a definite z-component are eigenstates.

To find the first order shift in the energies for these states, we can use ordinary perturbation theory for the states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. The energy shifts for these states are:

$$\delta E'_{\uparrow\uparrow} = \langle\uparrow\uparrow|S_x^{(1)}S_x^{(2)}|\uparrow\uparrow\rangle$$
$$= \left(\frac{\hbar}{2}\right)^2\langle\uparrow\uparrow|\downarrow\downarrow\rangle$$
$$= 0$$

we are using that the matrix for S_x is $\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and

$$\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

Similarly, we find $\delta E'_{\downarrow\downarrow} = 0$.

To find $\delta E_{\uparrow\downarrow}$ and $\delta E_{\downarrow\uparrow}$, we need degenerate perturbation theory. Let's call $|A\rangle = |\uparrow\downarrow\rangle$, $|B\rangle = |\downarrow\uparrow\rangle$

Then for small λ, the eigenstates of $H_0 + \lambda H_1$ with energies close to 0 will be close to some states $\alpha_{(1)}|A\rangle + \beta_{(1)}|B\rangle$ and $\alpha_{(2)}|A\rangle + \beta_{(2)}|B\rangle$.

To find these states and the corresponding energy shifts $\delta E_{(1)}$ and $\delta E_{(2)}$, we need to find the solutions to the eigenvalue problem

$$\begin{pmatrix} \langle A|H_1|A\rangle & \langle A|H_1|B\rangle \\ \langle B|H_1|A\rangle & \langle B|H_1|B\rangle \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \delta E \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

We find $\langle A|H_1|A\rangle = \langle B|H_1|B\rangle = 0$ and

$$\langle A|H_1|B\rangle = \langle \uparrow\downarrow | S_x^{(1)} S_x^{(2)} | \downarrow\uparrow\rangle = \left(\frac{\hbar}{2}\right)^2 \langle \uparrow\downarrow | \uparrow\downarrow\rangle = \left(\frac{\hbar}{2}\right)^2$$

$$\langle B|H_1|A\rangle = \langle A|H_1|B\rangle^* = \left(\frac{\hbar}{2}\right)^2$$

So our matrix is $\left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This has eigenvector

$\begin{pmatrix} \alpha_{(1)} \\ \beta_{(1)} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with eigenvalue $\left(\frac{\hbar}{2}\right)^2$ and

$\begin{pmatrix} \alpha_{(2)} \\ \beta_{(2)} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalue $-\left(\frac{\hbar}{2}\right)^2$. So the state

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ gets a shift $\lambda\left(\frac{\hbar}{2}\right)^2$ to order λ, and

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ gets a shift $-\lambda\left(\frac{\hbar}{2}\right)^2$ to order λ. The full set of energies to this order look like: