Consider states

$$|n\ \ell\ m\rangle \otimes |s_z\rangle$$

of a hydrogen atom, with $\ell = 1$ and $n = 2$.

We'll denote these by $|m\rangle \otimes |s_z\rangle$.

a) Write each of the possible basis states, and for each state, write the eigenvalue of

$$J_z = L_z + S_z$$

state:	J_z
$\|1\rangle \otimes \|\frac{1}{2}\rangle$	$\frac{3}{2}$
$\|1\rangle \otimes \|-\frac{1}{2}\rangle$	$\frac{1}{2}$
$\|0\rangle \otimes \|\frac{1}{2}\rangle$	$\frac{1}{2}$
$\|0\rangle \otimes \|-\frac{1}{2}\rangle$	$-\frac{1}{2}$
$\|-1\rangle \otimes \|\frac{1}{2}\rangle$	$-\frac{1}{2}$
$\|-1\rangle \otimes \|-\frac{1}{2}\rangle$	$-\frac{3}{2}$

b) These states aren't eigenstates of total angular momentum J^2. But some linear combinations of them are eigenstates of J^2 and J_z (label these by $|J\ M\rangle$). Based on the values of J_z you found, and the fact that states with some J always come in groups with $M = -J, -J+1, \ldots, J$, what are the possible values of (J, M)?

Have groups $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ and $-\frac{1}{2}, \frac{1}{2}$ for J_z, corresponding to $J = \frac{3}{2}$ and $J = \frac{1}{2}$

c) Each $|J\ M\rangle$ state is a linear combination of $|m\rangle \otimes |s_z\rangle$ states, but two of the states are just equal to a single $|m\rangle \otimes |s_z\rangle$ state. Which are they?

$$|\tfrac{3}{2}\ \tfrac{3}{2}\rangle = |1\rangle \otimes |\tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle = |-1\rangle \otimes |-\tfrac{1}{2}\rangle$$

since no other states have $m + s_z = \pm\frac{3}{2}$

d) Starting from the one with the larger M, act on the equation $|J\ M\rangle = |m\rangle \otimes |s_z\rangle$ with $J_- = L_- + S_-$ (use J_- on the left and $L_- + S_-$ on the right) to determine how to write the other $|J\ M\rangle$ states for this J.

$$J_-|\tfrac{3}{2}\ \tfrac{3}{2}\rangle = \sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2} - \tfrac{3}{2}\cdot\tfrac{1}{2}}\ |\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{3}\ |\tfrac{3}{2}\ \tfrac{1}{2}\rangle$$

also $J_- = L_- + S_-$ and $|\tfrac{3}{2}\ \tfrac{3}{2}\rangle = |1\rangle \otimes |\tfrac{1}{2}\rangle$ so

$$\begin{aligned}
\sqrt{3}\ |\tfrac{3}{2}\ \tfrac{1}{2}\rangle &= (L_- + S_-)|1\rangle \otimes |\tfrac{1}{2}\rangle \\
&= (L_-|1\rangle) \otimes |\tfrac{1}{2}\rangle + |1\rangle \otimes (S_-|\tfrac{1}{2}\rangle) \\
&= \sqrt{2}\ |0\rangle \otimes |\tfrac{1}{2}\rangle + |1\rangle \otimes |-\tfrac{1}{2}\rangle
\end{aligned}$$

Thus: $|\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\ |0\rangle \otimes |\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\ |1\rangle \otimes |-\tfrac{1}{2}\rangle$

Similarly, we get $J_-|\tfrac{3}{2}\ \tfrac{1}{2}\rangle = 2|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle$ so

$$\begin{aligned}
2|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}}(L_-|0\rangle) \otimes |\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\ |0\rangle \otimes (S_-|\tfrac{1}{2}\rangle) + \sqrt{\tfrac{1}{3}}(L_-|1\rangle) \otimes |-\tfrac{1}{2}\rangle \\
&= \frac{2}{\sqrt{3}}\ |-1\rangle \otimes |\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\ |0\rangle \otimes |-\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\ |0\rangle \otimes |-\tfrac{1}{2}\rangle
\end{aligned}$$

$$\Rightarrow |\tfrac{3}{2}\ -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\ |-1\rangle \otimes |\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\ |0\rangle \otimes |-\tfrac{1}{2}\rangle$$

Finally, we already have $|\tfrac{3}{2}\ -\tfrac{3}{2}\rangle = |-1\rangle \otimes |-\tfrac{1}{2}\rangle$

e) Can you find the remaining $|J\ M\rangle$ states using the fact that they must be orthogonal to the ones you found?

We know that $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle = A\,|1\rangle\otimes\left|-\frac{1}{2}\right\rangle + B\,|0\rangle\otimes\left|\frac{1}{2}\right\rangle$

since the basis states on the right are the only ones with $L_z+S_z=\frac{1}{2}$.

$\left|\frac{1}{2}\ \frac{1}{2}\right\rangle$ should also be orthogonal to

$$\left|\frac{3}{2}\ \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\,|0\rangle\otimes\left|\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\,|1\rangle\left|-\frac{1}{2}\right\rangle$$

and we can assume $|A|^2+|B|^2=1$ for normalization, so (up to a phase),

$$\left|\frac{1}{2}\ \frac{1}{2}\right\rangle = -\sqrt{\frac{1}{3}}\,|0\rangle\otimes\left|\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\,|1\rangle\otimes\left|-\frac{1}{2}\right\rangle$$

Acting with $J^- = L^- + S^-$, we find

$$J^-\left|\frac{1}{2}\ \frac{1}{2}\right\rangle = -\sqrt{\frac{1}{3}}\,L^-|0\rangle\otimes\left|\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\,|0\rangle\otimes S^-\left|\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\,L_-|1\rangle\otimes\left|-\frac{1}{2}\right\rangle$$

$$\Rightarrow \quad \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\,|0\rangle\otimes\left|-\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\,|-1\rangle\otimes\left|\frac{1}{2}\right\rangle$$

We can also check that this is orthogonal to $\left|\frac{3}{2}\ -\frac{1}{2}\right\rangle$.