Bell's Inequalities

Imagine an experiment where two experimenters, Alice and Bob, are each fed a particle upon which they can perform one of two measurements. For her particle, Alice measures a quantity Q or a quantity R, each of which can be ± 1, depending on the state of the particle. Bob measures either a quantity S or a quantity T, each of which can be ± 1, depending on the state of his particle. For example, the quantities Q,R,S, and T could be ($2/\hbar$ times) the spin of the particle in different directions.

Now, let's assume that the particles' properties are fully determined prior to the measurement, and that Alice's measurements do not influence those of Bob's. We assume that each time Alice and Bob receive their particles, there is some probability $p(P, Q, R, S)$ that Alice's particle has properties P and Q, while Bob's has properties R and S.

a) Show that for any of the allowed combinations (P, Q, R, S),

$$QS + RS + RT - QT = \pm 2 \tag{1}$$

Hint: there are only 16 possibilities. For a fancier proof, you can use that $QS + RS + RT - QT = (Q + R)S + (R - Q)T$

b) Suppose Alice measures Q and Bob measures S. Define $\langle QS \rangle$ to be the average of the product of Q and S if these measurements are carried out for a large number of particle pairs assuming the probability distribution $p(P, Q, R, S)$. Thus,

$$\langle QS \rangle = \sum_{P,Q,R,S} p(P, Q, R, S) QS \tag{2}$$

Using the result above, show that

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle \leq 2 \; . \tag{3}$$

b) Now consider a quantum mechanical experiment where Alice and Bob receive particles whose spins are entangled, each time being in the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle) \; . \tag{4}$$

Let $Z \equiv \frac{2}{\hbar} S_z$ and $X \equiv \frac{2}{\hbar} S_x$ so that Z and X have possible values ± 1 in a measurement. Define $Q = Z_1$, $R = X_1$, $S = -(X_2 + Z_2)/\sqrt{2}$, and $T = (Z_2 - X_2)/\sqrt{2}$. By calculating quantum expectation values in the usual way, show that for the state (4), we have

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2} \tag{5}$$

This violates the inequality (3). Since experimental measurements give the quantum results, the assumptions that we used to derive the inequality (that the particles has predetermined measurement outcomes and that the measurements made by Alice and Bob don't influence each other) don't hold in nature! This shows that quantum mechanics can't be secretly just some ordinary theory where everything has a definite state before the measurement.