Tutorial #5: Mass, Energy, and Momentum

We have seen that in order to keep conservation of momentum as a valid physical law, it is necessary to modify the definition of momentum to:

$$\vec{p} = m\frac{d\vec{x}}{d\tau} = m\gamma\vec{u} \qquad \gamma = \frac{1}{\sqrt{1-\frac{|\vec{u}|^2}{c^2}}}$$

where $\vec{u}$ is the velocity of the object and m is its mass.

With this definition, the total momentum of all objects in the system is independent of time (i.e. conserved) in any frame of reference.

Question 1

Two objects collide with a third object (initially at rest) of mass M. If these two objects are at rest after the collision, while the third object is observed to have velocity components u_x=16/25c and u_y=12/25c, what were the momenta for the moving objects before the collision? (answer in terms of M and c)

A:

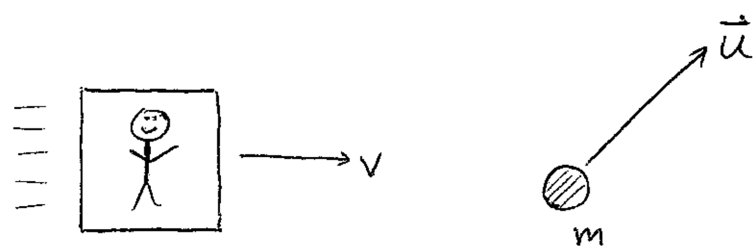
B:

If object B was initially observed to be moving at speed 4/5c, what is its mass?

$M_B =$

Question 2

Conservation of momentum does not mean that momentum is the same in all frames of reference. For example, if an object is at rest in one frame of reference (so momentum is zero), it will be moving and therefore have momentum in any other frame of reference. So a natural question to ask is how the momentum measured in different frames of reference is related. We can work this out using the Lorentz Transformation and the definition of momentum above.



Suppose we have an object with velocity u and we want to find its momentum in the frame of an observer moving at velocity v in the +x direction. Let's say that in a time Δt in the original frame, the object is observed to move a distance Δx in the x

direction, Δy in the y direction, and Δz in the z direction. Call the proper time that passes during this time $\Delta\tau$. The momentum in the original frame is then:

$$p_x = m\frac{\Delta x}{\Delta\tau} \qquad p_y = m\frac{\Delta y}{\Delta\tau} \qquad p_z = m\frac{\Delta z}{\Delta\tau}$$

On the other hand, the momentum for the new observer will be:

$$p_x' = m\frac{\Delta x'}{\Delta\tau} \qquad p_y' = m\frac{\Delta y'}{\Delta\tau} \qquad p_z' = m\frac{\Delta z'}{\Delta\tau}$$

i.e. mass times the change in position in her coordinates over the change in proper time.

a) Calculate the following quantities in the new frame in terms of times and distances in the original frame:

$\Delta x' =$

$\Delta y' =$

$\Delta z' =$

b) Argue that the y and z momentum in the new frame are the same as in the original frame

c) Using your results above, you should find that

$$p'_x = \gamma \,(p_x - v/c^2\, E) \qquad (\star)$$

where E is some quantity involving m, c, Δt, and $\Delta\tau$. What is E?

E =

d) How is Δt related to $\Delta\tau$? If your answer involves γ, make sure to specify which velocity γ should be calculated with.

e) Using your answers for parts c and d, express the quantity E in terms of m, c, and the velocity,

E =

The equation $(\star)$ above shows that the x momentum in the new frame of reference is determined in terms of the x momentum in the original frame and the quantity E we have just calculated. Conservation of momentum tells us that the sum of p'_x for all objects is the same before and after any collision, and also that the sum of p_x for all objects is the same before and after any collision. If equation $(\star)$ is correct, it must also be that the sum of E for all objects is the same before and after any collision. So we have found a new conserved quantity! We can check that E has units of energy, so we will define it to be the RELATIVISTIC ENERGY of an object. Before proceeding, check your answer for e) with me or a TA to make sure that you have the right formula for relativistic energy.

Question 3

Using the approximation $\frac{1}{\sqrt{1-x^2}} \approx 1 + \frac{1}{2}x^2$ for small x, show that for an object with small velocity, we have

$$E \approx (\text{Mass} \times c^2) + (\text{Kinetic Energy})$$

In ordinary mechanics, mass is always conserved while kinetic energy is conserved in elastic collisions. What we have found here is that some combination of mass and kinetic energy is conserved in all relativistic collisions. However, we have *not* shown that they each have to be conserved separately. As we are about to discover, if momentum and relativistic energy are conserved, then mass *cannot* be a conserved quantity.

Question 4

Suppose that we have two objects of mass *m* that travel towards each other, each with speed v = 4/5c.

a) What is the total relativistic energy before the objects collide (answer in terms of *m* and c)?

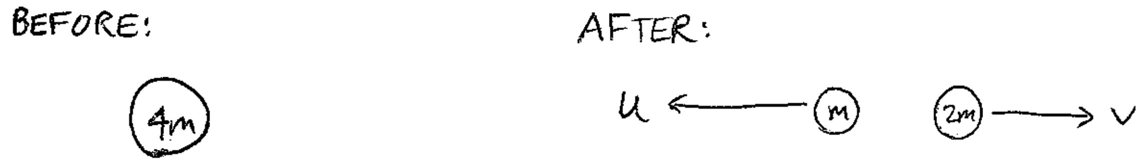
b) If the objects stick together when they collide to form a new object, what is the velocity of the object after the collision? Why?

c) If the mass of the new object is M, what is the total relativistic energy after the collision (in terms of M and c)?

d) Assuming energy is conserved, what is M in terms of m? Is mass conserved here?

Question 5

A nucleus of mass 4m decays into two smaller nuclei of mass m and 2m:



a) Using momentum conservation, determine an equation relating the speeds u and v of the two particles.

b) Using energy conservation, determine another equation relating the two speeds.

Since we now have two equations for two unknowns, we can in principle solve for the two speeds. You don't need to do that here, but the point is to recognize that conservation of energy and momentum allow us to completely predict the outgoing velocities of the two particles in this radioactive decay (without knowing anything about nuclear physics!).

Question 6

In a constant electric field, the force on a charged particle is

$$F = Eq$$

If we use the right definition of momentum, then Newton's 2nd Law still holds. Determine the trajectory x(t) for a particle that starts at rest in a constant electric field. What happens to the velocity as t goes to infinity?