Last time in Phys 157...
Exponential decay:
- when amplitude/energy decreases by fixed fraction each period

\[ A = A_0 e^{-t/t_0} \]

- time constant = time before amplitude is \( \frac{1}{e} \times \) initial amplitude

\[ A = A_0 e^{-1} = 0.368 A_0 \]

\[ A = A_0 e^{-2} \]
Damped oscillations

amplitude:

\[ A = A_0 e^{-\frac{t}{t_0}} \]

multiplies cosine

\[ x(t) = A_0 e^{-\frac{t}{t_0}} \cos(\omega t + \phi) \]
The graph shows displacement vs time for a damped oscillation. The time constant $t_0$ in this case is nearest to

A) 1s    B) 3s    C) 5s    D) 7s    E) 9s

**EXTRA:** Can you find $t_0$ exactly?
The graph shows displacement vs time for a damped oscillation. The time constant \( t_0 \) in this case is nearest to

A) 1s  
B) 3s  
C) 5s  
D) 7s  
E) 9s

EXTRA: Can you find \( t_0 \) exactly?

At circled points, \( \cos = 1 \) so \( x(t) = A_0 e^{-t/t_0} \)

At \( t = 0 \), \( x = 8 \text{cm} \). At \( t = 2 \text{s} \), \( x = 6 \text{cm} \).

\[ 6 \text{cm} = 8 \text{cm} \times e^{-\frac{2}{t_0}} \]

\[ e^{-\frac{2}{t_0}} = 0.75 \]

\[ -\frac{2}{t_0} = \ln(0.75) \]

\[ t_0 \approx 7 \text{s} \]
Forces that lead to damping are velocity dependent & opposite direction to velocity.

**Examples:**

- **Friction:** 
  - $F_f$  
  - $v$

- **Drag forces in air or fluids:**
  - $F_D$
  - $v$
Example: drag forces from air/fluids

\[ F_{\text{drag}} \]

\[ V \]

\[ F_{\text{drag}} \]

small \( V \): \( F = -bv \)

large \( V \): \( |F| \propto V^2 \)
Example: viscous fluid drag

$F_D = -bv$

$F_{NET} = -kx - bv$

Equations of motion:

\[
\frac{dx}{dt} = v
\]

\[
\frac{dv}{dt} = -\frac{k}{m} x - \frac{b}{m} v
\]

This is $a = \frac{F}{m}$

Use these to predict how $x$ and $v$ change with time.
\[
\frac{dx}{dt} = v \\
\frac{dv}{dt} = -\frac{k}{m} x - \frac{b}{m} v
\]

Solution is:

\[
x(t) = A_o e^{-\frac{t}{t_o}} \cos(\omega t + \phi)
\]

\[
t_o = \frac{2m}{b} \\
\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}
\]

Check: calculate \( v = \frac{dx}{dt} \) and then verify 2nd eqn.

Valid for \( b < 2\sqrt{km} \)
Simulation or demo of damped oscillation
no damping (idealized situation)

\[ b = 0 \]
\[ b = 0.1 \times 2\sqrt{km} \]

still have \( \omega \approx \omega_{b=0} = \sqrt{\frac{k}{m}} \)
\[ b = 0.5 \times 2\sqrt{k} \text{m} \]
Critical damping

\[ b = 2\sqrt{k} \quad \Rightarrow \quad \omega = 0 \quad \text{pure decay, no oscillations} \]
Overdamping: \( b > 2\sqrt{km} \)

also exponential decay, but slower to reach equilibrium than critical damping
An object with mass 2kg oscillates on a spring with a small amount of damping.

a) What is the damping constant $b$?

EXTRA: What is the spring constant $k$?
An object with mass 2kg oscillates on a spring with a small amount of damping.

a) What is the damping constant $b$?

EXTRA: What is the spring constant $k$?

\[
A = A_0 e^{-t/\tau_0} \quad \Rightarrow \quad 1.55\text{cm} = 2\text{cm} \cdot e^{-2sA_0} \\
0.775 = e^{-2s} \\
\tau_0 = \frac{-2s}{\ln(0.775)} = 7.85s \\
\]

\[
\text{step 2: find } b:\ \\
t_0 = \frac{2m}{b} = \frac{2 \cdot 2}{b} = \frac{4\text{kg}}{7.85s} \\
= 0.51 \frac{\text{kg}}{s}
\]
An object with mass 2kg oscillates on a spring with a small amount of damping.

a) What is the damping constant $b$?

EXTRA: What is the spring constant $k$?

accurate to just use $w = \sqrt{\frac{k}{m}}$ unless highly damped.

step 1: find $\omega$ $T = 2s$ so $\omega = \frac{2\pi}{T} = 3.1s^{-1}$

step 2: use $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

we can ignore these

$\Rightarrow k = m\omega^2 + \frac{b^2}{4m}$

$= 19.2\frac{N}{m} + 0.03\frac{N}{m} \approx 19.2\frac{N}{m}$
Forced Oscillations:

We can add in an oscillating force by hand:

\[ F = F_0 \cos(\omega_D t) \]

- Object will end up oscillating at driving frequency but amplitude largest if \( \omega_D \) matches \( \omega_0 = \sqrt{\frac{k}{m}} \)

- This is Resonance
Objects with the masses shown each sit in equilibrium on different springs, all with spring constant 200 N/m and damping constant 1 kg/s. If we turn on a driving force with frequency $f = 1.59 \text{ s}^{-1}$, which mass will oscillate with the largest amplitude?

A) 1  
B) 2  
C) 3  
D) 4  
E) 5
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C) 3
D) 4
E) 5

\[
\text{Have } f_0 = 1.59\text{ s}^{-1} \quad \Rightarrow \quad \omega_0 = 2\pi f_0 \approx 10\text{ s}^{-1}
\]

Want this to match
\[
\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\text{ N/m}}{m}}
\]
so \( m = 2\text{ kg} \) works.