Chapter 21: **Superposition**

Section 21.1

Principle of Superposition

If two or more travelling waves are moving through a medium, the resultant wave function at any point is the algebraic sum of the wave functions of the individual waves.

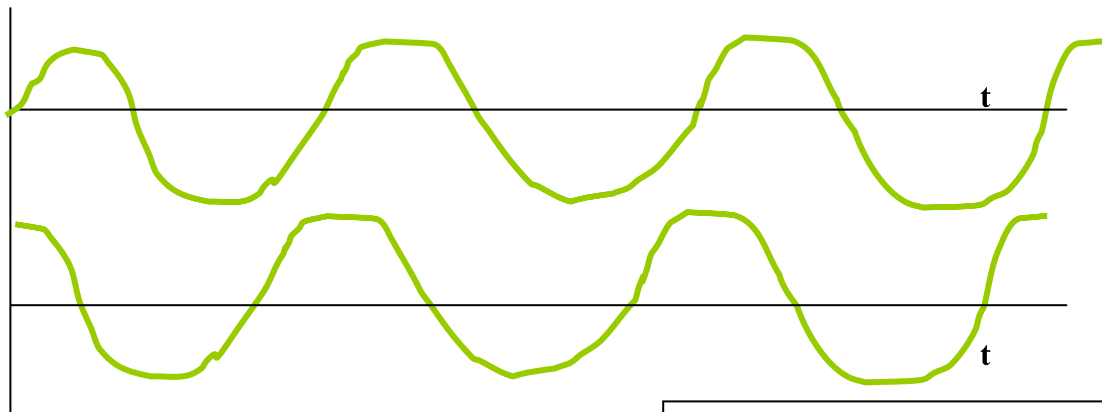
The principle of superposition depends on the medium having a linear response to perturbations. I.e. Hookes Law, F = -kx. For an elastic medium, the linear regime is defined as that where Hookes law is obeyed.

Interference: when waves add, the interference is constructive. When waves cancel each other, the interference is destructive.

Consider two sinusoidal travelling waves with the same wavelength and velocity but different phases.

D_1

D_2



$D_1 = D_M \sin(kx - \omega t)$ **$D_2 = D_M \sin(kx - \omega t + \phi)$**

Note that D_2 is just like D_1 but it has a relative phase difference of ϕ. The text uses $\Delta\phi$

We wish to find the sum $D_1 + D_2$

First, we need the trigonometric identity:

I am assuming that everyone is familiar with this basic identity.

$\sin(a) + \sin(b) = 2\cos[(a - b)/2]\, \sin[(a + b)/2]$

To prove this identity, use **$\sin(\theta_1 + \theta_2) = \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2$**

Then set **$\theta_1 = (a - b)/2$** and **$\theta_2 = (a + b)/2$**, to get

$\sin(a) = \cos[(a-b)/2]\, \sin[(a+b)/2] + \sin[(a-b)/2]\, \cos[(a+b)/2]$ Eqn #1

Followed by **$\theta_1 = (b-a)/2$** and **$\theta_2 = (b+a)/2$**, to get

sin(b) = cos[(b-a)/2] sin[(b+a)/2] + sin[(b-a)/2] cos[(b+a)/2] Eqn #2

Now, **cos[(b-a)/2] = cos[(a-b)/2]** and **sin[(b-a)/2] = - sin[(a-b)/2]**

So, summing eqns 1 and 2 we get:

And, of course, cos(a+b) = cos(b+a) and sin(a+b) = sin(b+a).

sin(a) + sin(b) = 2cos[(a-b)/2] sin[(a+b)/2]

Now we can use the above with $a = kx - \omega t$ and $b = kx - \omega t + \phi$ to get:

$$D = D_1 + D_2 = 2D_M\cos(\phi/2)\sin(kx - \omega t + \phi/2)$$

Note that the sine part of the equation is the original sinusoidal wave with wave number k and angular frequency v. Its phase angle is 1/2 the relative phase angle between the two waves being added. The amplitude is given by $2D_M\cos(\phi/2)$.

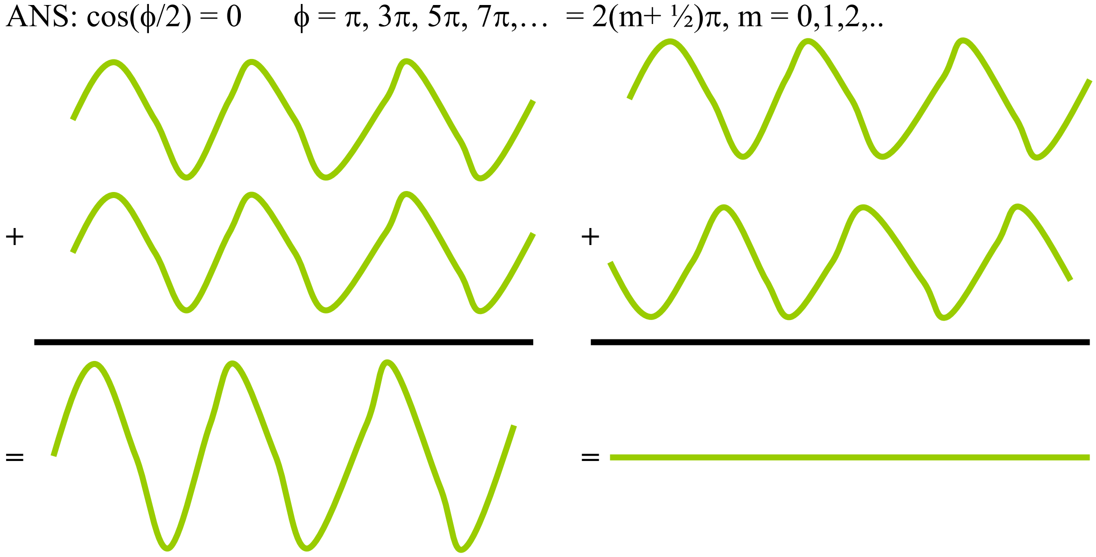
For what relative phases ϕ do we get constructive interference?

ANS: $\cos(\phi/2) = \pm 1$ $\phi = 0, 2\pi, 4\pi, 6\pi, \ldots = 2m\pi$, $m = 0,1,2,3,\ldots$

Question: When $\cos(\phi/2)$ is -1, is the resultant wave inverted with respect to D_1?
ANS: no {note: if $\phi/2 = 180$, $\sin(\theta + \frac{1}{2}\phi) = \sin\theta \cos\pi = -\sin\theta$, so $D = 2D_M\sin(kx-\omega t)$} And of course, this corresponds to a 2π phase shift!

For what relative phases ϕ do we get destructive interference?

ANS: $\cos(\phi/2) = 0$ $\phi = \pi, 3\pi, 5\pi, 7\pi, \ldots = 2(m+\frac{1}{2})\pi$, $m = 0,1,2,\ldots$

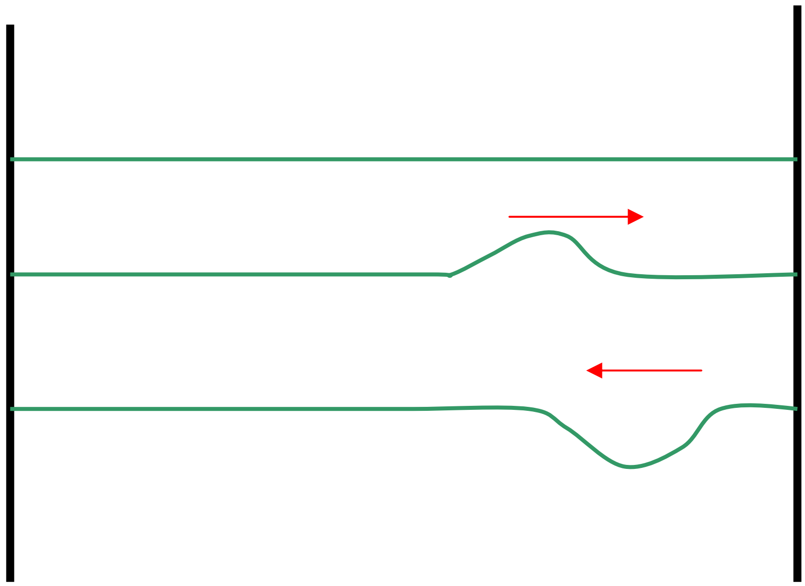


$\cos(\phi/2) = \pm 1$ $\cos(\phi/2) = 0$

$\phi = 0, 2\pi, 4\pi, \ldots = m(2\pi)$, $m=0,1,2\ldots$ $\phi = \pi, 3\pi, 5\pi, \ldots = 2(m+\frac{1}{2})\pi$, $m=0,1,2\ldots$

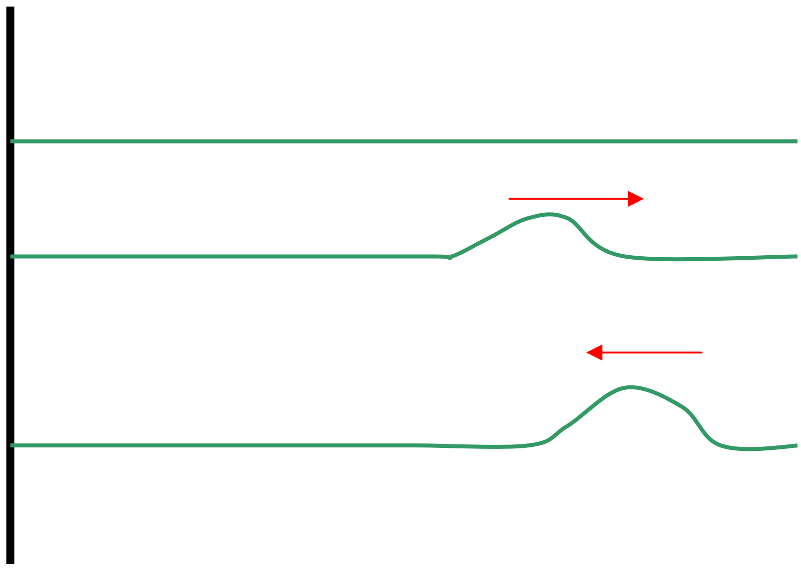
Sections 21.2, 3, 4 Standing waves, transverse standing waves and standing sound waves and musical acoustics.

Reflection: Consider a string fixed to a solid wall at both ends:



If we send a wave pulse down the string it will be reflected and inverted at the wall. This amounts to a phase change of 180^o at the wall boundary. The inversion is caused by the reaction force of the wall against the string as the wave is pulled down when it meets the wall.

Consider another string which moves freely at its right end.



This diagram might look more plausible if it were turned 90^o so the rope was hanging vertically down.

If we send a wave pulse down this string, the wave pulse will be reflected but not inverted at the right hand end.

The action of sound waves in pipes follows an analogous trend.

StringWave Simulation

Demonstrate what happens to a pulse when it gets to a fixed end and when it gets to a loose end.

http://phet.colorado.edu/simulations/stringwave/stringWave.swf

1) Fixed both ends: Use the Oscillate mode with the fixed end. Set Tension high, damping 0, and amplitude 20.

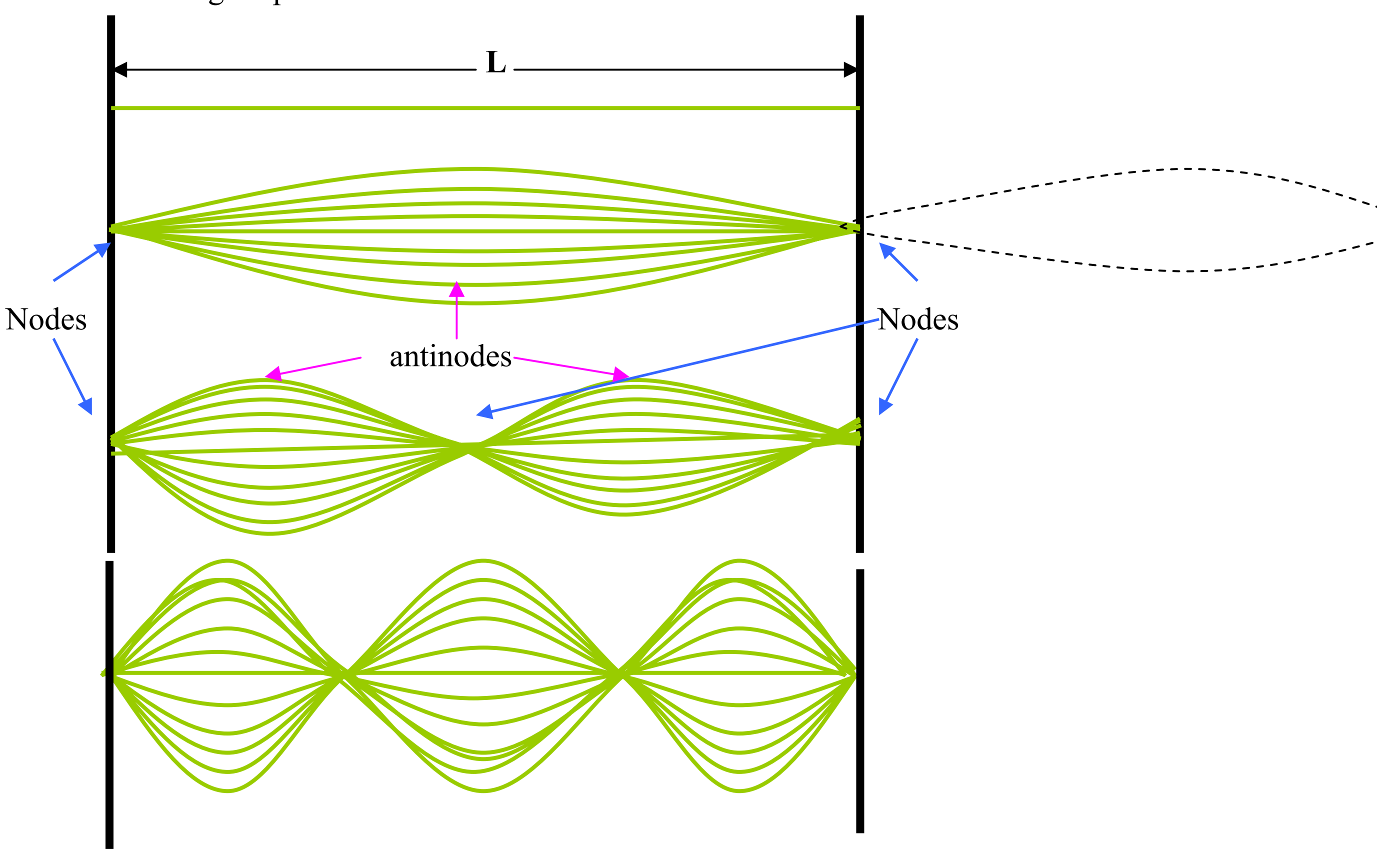
Set frequency 8 for lowest mode
Set frequency 16 for next mode
Set frequency 25 for next mode next 33, next 42,

2) Fixed one end: Same as above but one end loose

Set frequency 5 for lowest mode
Set frequency 13 for next mode
Set frequency 21 for next mode, next 30

Standing waves

Consider a string suspended between two fixed walls:



If we excite it at random, chances are that reflected waves will interfere with each other and the disturbance will dissipate. However, if we excite it at a particular frequency which depends on the length of the string, we can excite a resonance or standing wave.

The standing wave is the sum of waves. We define **nodes** where the waves interfere destructively to produce zero amplitude and **antinodes** where the waves interfere constructively to produce maximum amplitude.

Standing waves occur at more than one frequency. Allowed frequencies are determined by the placement of nodes and antinodes.

For a string fixed at both ends, the longest wavelength occurs when there are nodes at each end and one antinode in the centre. Then $\lambda_1 = 2L$ and $f_1 = v/\lambda_1$. This is called the fundamental frequency. The next higher frequencies are called the second and third harmonics and they occur with two and three antinodes for $\lambda_2 = L$ and $\lambda_3 = 2/3L$.

The fundamental frequency is sometimes called the first harmonic. The higher harmonics are sometimes called overtones.

The harmonics are characterised by $\boldsymbol{\lambda_m = 2L/m \quad m = 1,2,3,\ldots.}$

The natural frequencies are $\boldsymbol{f_m = v/\lambda_m = \frac{mv}{2L}}$ recall that for a string, $\boldsymbol{v = \sqrt{(T_s/\mu)}}$

Then $\boldsymbol{f_m = \frac{m}{2L}\sqrt{(T_s/\mu)}}$ are the frequencies of the harmonics of this system.

This is how a guitar works. The thicker strings play the lowest frequency music. If plucked in the middle, the first harmonic is the main mode excited. Because the string is plucked near one end and not the middle, more than one harmonic is excited in the standing waves. To change the frequency of the standing waves, the guitar player moves their finger up and down the strings to change the length of available string. To tune the instrument, the player adjusts the tension in the strings.

Mathematical representation of a standing wave

A standing wave is actually made up of two travelling waves travelling in opposite directions one with velocity v (to the right) and the other with velocity -v (to the left).

What does this say about energy? An equal amount of energy is being transferred in each direction. Standing waves store energy.

The two waves are $\boldsymbol{D_1(x,t) = D_M\sin(kx - \omega t)}$ and $\boldsymbol{D_2(x,t) = D_M\sin(kx + \omega t)}$.

$$\boldsymbol{D = D_1 + D_2 = D_M[\sin(kx - \omega t) + \sin(kx + \omega t)]}$$

Use $\mathbf{\sin(\theta_1) + \sin(\theta_2) = 2\sin\frac{1}{2}(\theta_1 + \theta_2)\cos\frac{1}{2}(\theta_1 - \theta_2)}$

And $\cos(-\varphi) = \cos(\varphi)$

Derived before

Then $\mathbf{D = 2D_M \sin(kx)\cos(\omega t)}$

This equation tells us that the string oscillates at the angular frequency ω everywhere but has an amplitude which varies with x as $2D_M\sin(kx)$.

We know that x=0 and x=L must correspond to nodes, so $kL = m\pi$ where $k=2\pi/\lambda$

Then we can write $\lambda = 2L/m$

This means that the fixed points of the string at each end determine the wavelengths of possible standing waves.

The nodes occur where $\sin(kx) = 0$ or

$kx = m\pi$ or $(2\pi/\lambda)x = m\pi$, Then $x = m\lambda/2$, m=1,2,3

Sine waves pass through the origin every half wavelength. Each mode has a different λ.

The antinodes occur when $\sin(kx) = \pm 1$ or where $kx = m\pi/2$, m=1,3,5,..

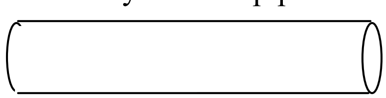
Or $x = m\lambda/4$, m=1,3,5..

Note that here we deal only with positive m.

Standing waves in air columns

When we discussed standing waves, we considered a system where the string is attached at both ends. For air column, this type of system is not useful since there is no way for the sound to leave the column. In air columns, we look at two types of system: one with both ends open and the other with one end open.

Consider a cylindrical pipe closed at one end:



Standing longitudinal sound waves can be set up in this column. For sound waves there are two variables:

1) **Displacement:** $\mathbf{D(x.t) = D_M \sin(kx - \omega t)}$
   Note that this displacement is along the longitudinal direction

2) **Pressure:** $\mathbf{\Delta P = -\Delta P_M \cos(kx - \omega t)}$

When we talked about nodes earlier, we meant displacement nodes. Now we can talk about pressure

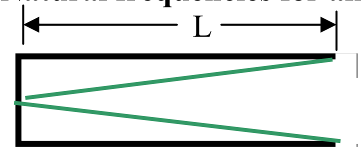
The closed end of an air column is a displacement node and a pressure antinode.

Air molecules cannot move through a solid wall. Pressure builds up (and dies down) as the air molecules push (and pull) against the wall.

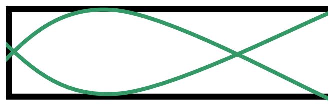
The open end of an air column is a displacement antinode and a pressure node.

Pressure at an open end is fixed at atmospheric pressure.

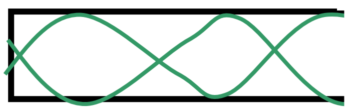
Natural frequencies for air columns



$L = \lambda/4$ Fundamental $\lambda = 4L$

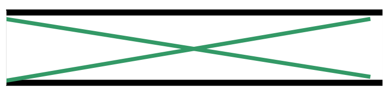


$L = 3\lambda/4$ Third Harmonic $\lambda = 4/3L$

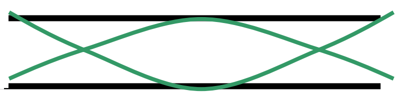


$L = 5\lambda/4$ Fifth Harmonic $\lambda = 4/5L$

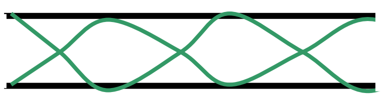
For a pipe **closed at one end**, $f_m = \dfrac{mv}{4L}$ $m = 1,3,5,..$ v = velocity of sound, $v = \lambda f$



$L = \lambda/2$ Fundamental $\lambda = 2L$



$L = \lambda$ Second harmonic $\lambda = 2L/2$

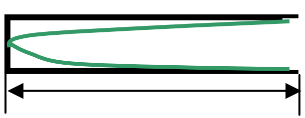


$L = 3/2\lambda$ Third harmonic $\lambda = 2L/3$

For a pipe **open at both ends**, we get: $f_m = \dfrac{mv}{2L}$ $m = 1,2,3,\ldots$

Problem: Find the fundamental frequency for your eardrum.

In the average adult, the auditory canal is about 1 ml in volume and 2.7 cm long. It can be approximated by a pipe with one end closed, the ear drum, and the other end open, the outer ear.



L = 2.7 cm

From the above relations $f_1 = v/4L = (343m/s)/[4(0.027m)]$
$f_1 = 3175$ Hz.
This frequency is quite close to the frequency of speech,
In fact, some claim that it is the frequency of a babies cry.

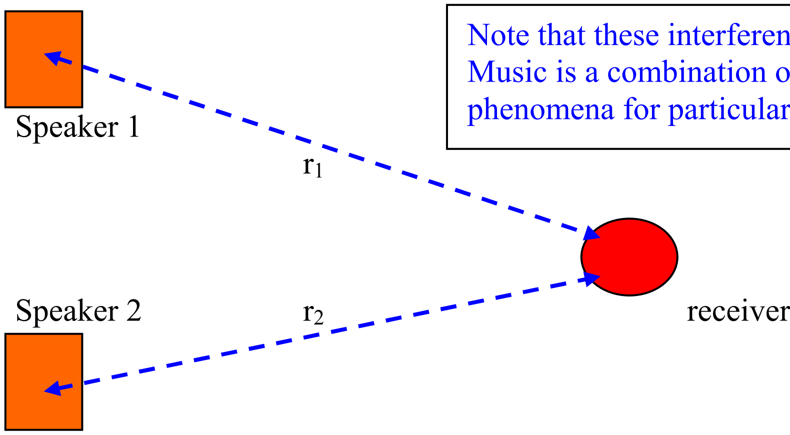
The auditory canal acts as a filter since it will set up a standing wave at the fundamental frequency. Small perturbations at this frequency will be successful in producing audible sounds.

Sections 21.5, 6, 7 Interference of waves

Superposition of waves of the same frequency produces <u>spatial</u> interference

Example: Interference of sound waves

Consider a sound system containing two speakers:



Note that these interference phenomena occur at discrete frequencies. Music is a combination of many frequencies so the interference phenomena for particular frequencies are much harder to detect.

We use r rather than Δx (as in the text) to make this a 3D problem as in section 21.7.

Speaker's 1 and 2 transmit identical sound waves of wavelength λ and relative phase ϕ zero.

What are the conditions for r_1 and r_2 such that there is constructive (or destructive) interference at the receiver?

Note that for each speaker, $D = D_M \sin(kr - \omega t)$; where r is the distance between the speaker and the listener. At any time t, the relative phases are determined by the relative distances from each speaker; i.e. the phase difference between the speakers is $kr_1 - kr_2$ or $2\pi r_1/\lambda - 2\pi r_2/\lambda$. The relative phase difference $\Delta\phi$ between speakers 1 and 2 is then

$$\boldsymbol{\Delta\phi = 2\pi(r_1 - r_2)/\lambda} \quad \textbf{or equivalently, } \boldsymbol{r_1 - r_2 = \Delta\phi(\lambda/2\pi)}$$

Note that as $r_1 - r_2$ varies from 0 to λ, $\Delta\phi$ varies from zero to 2π.

We have already learned that (maximum) constructive interference occurs for $\Delta\phi = 2m\pi$, so we get constructive interference when

$$\boldsymbol{r_1 - r_2 = \Delta\phi(\lambda/2\pi) = (2m\pi)\lambda/2\pi = m\lambda} \textbf{ where } \boldsymbol{m = 0,1,2,3}$$

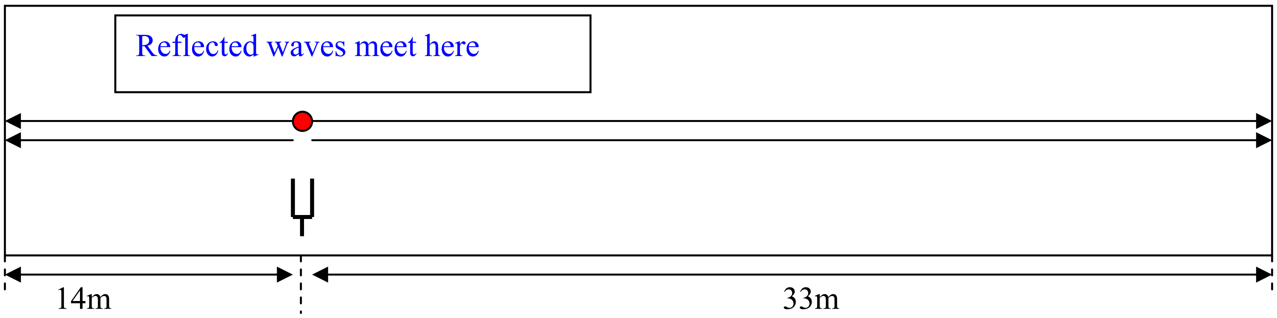
Similarly, for (perfect) destructive interference, we use $\Delta\phi = (m+ ½)2\pi$ (m= 0,1,2,3) to get

$$\boldsymbol{r_1 - r_2 = (m+ ½)\lambda}, \textbf{ where } \boldsymbol{m=0,1, 2,3,.}$$

The above relations assumed that the inherent phase of the speakers were the same. (i.e. at $r_n=0$, t=0, phase =0). If this is no the case we add a term $\Delta\phi_o$ to $\Delta\phi$ and

$$\boldsymbol{\Delta\phi = 2\pi(r_1 - r_2)/\lambda + \Delta\phi_o}$$

Problem: A tuning fork generates sounds with f = 246 Hz. The waves travel in opposite directions along a hallway, are reflected by walls at each end and return. The hall is 47.0 m long and the tuning fork is located 14 m from one end. What is the phase difference between the reflected waves when they meet back at the tuning fork? v = 343m/s



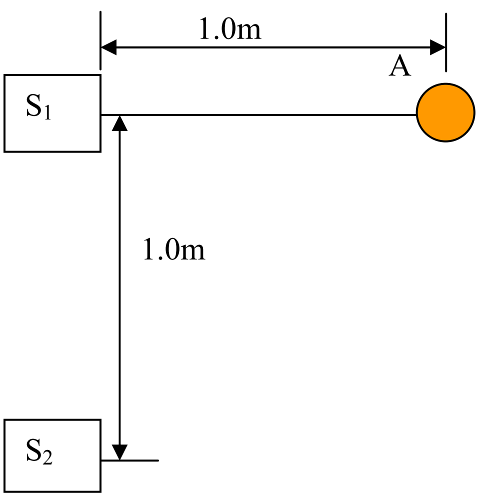
Then r_1 (to the left and back) is (47 -33)m x 2 = 28m

And r_2 = 66m

To put this problem another way, r_1-r_2 = 38m = 27.2536λ where λ = 1.39m. So, 38m = 27λ + 0.2536λ

$\phi = 2\pi(r_1 - r_2)/\lambda = 2\pi(r_1 - r_2)f/v = 2\pi(38\text{m})(246\ \text{s}^{-1})/(343\text{m/s}) = 2\pi(27.2536) = 2\pi(27+.2536) =$ *(171.24 rad)* = 0.2536(2π) = 1.59 rad = 91.39 degrees(multiply by 360/2π to get degrees)

Problem from midterm: Two speakers are attached to the same amplifier so that at each box they emit sound waves at the same phase. What is the lowest frequency that will provide constructive interference at the location of the observer A?



Solve $r_2 - r_1 = m\lambda$ where λ = v/f and m = 0,1,2,3,…

The lowest frequency corresponds to the longest wavelength (since λ = v/f) and hence the lowest value of m. Since r_1 does not equal r_2, then m=1 is the next lowest candidate. This corresponds to $\lambda = r_2 - r_1 = \lambda = (\sqrt{2} - 1) = 0.4142\text{m}$

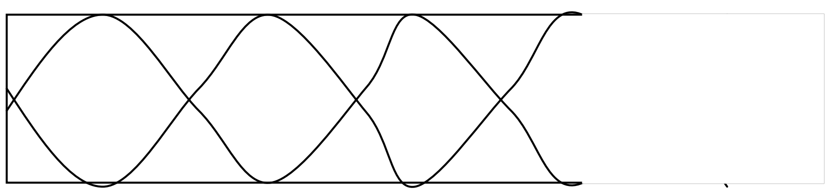
Then f = v/λ = (343m/s)/(0.414m) = 828 Hz

Problem from Dec 2004 Exam

A column of air in a tube is found to have standing waves at frequencies of 360, 504, and 648 Hz. There are no standing wave frequencies between the above frequencies.

(a) What is the fundamental frequency of the tube?
(b) Is the tube open at both ends or open at one and closed at the other?
(c) Draw a displacement curve for the 504 Hz standing wave.
(d) The air is now replaced with carbon dioxide which has a speed of sound of 280 m/s. What are the new frequencies corresponding to the given standing wave frequencies?

(a) For a tube closed one end, $f_m = mv/4L$, m=1,3,5.; for a tube open both ends, $f_m = mv/2L$, m = 1,2,3. and between adjacent modes, $\Delta f = v/2L$ in both cases. In this case, Δf = 144Hz. Note that the lower frequencies correspond to 216 Hz and 72 Hz. The lowest frequency, 72 Hz, corresponds to the fundamental frequency.
(b) If this were a tube open both sides, the fundamental frequency would have been 144Hz. So it is a tube open at one end
(c) The 504 Hz standing wave corresponds to the 4th possible frequency (m=7) or the 7th harmonic; There will be three anti-nodes.

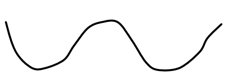


(d) Use $f_m = mv/4L$; For the fundamental mode, L = (343 m/s)/4×72Hz = 1.19m.

Then for CO_2, f_m' = m(280m/s)/(4×1.19m) = 58.8Hz, 176.5Hz, 294.1Hz and 411.7Hz for the 7th harmonic.

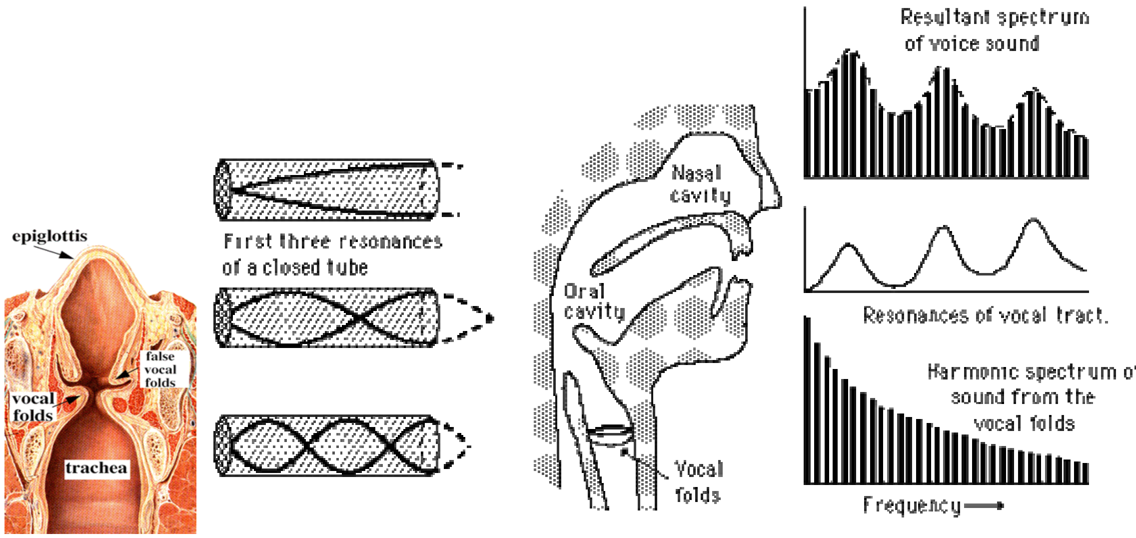
Speech and Hearing

Speech involves only a few moving parts: lungs, vocal cords, tongue, jaw, lips, and soft palate. The total capacity of our lungs is about 7 l. During quiet breathing, we use only about 0.5 l. During speech we use about 1.2 l, about 25% of our lung capacity. During speech, we are breathing out in sawtooth fashion with 95% of the time spent breathing out.

Normal Breathing  Breathing while speaking 

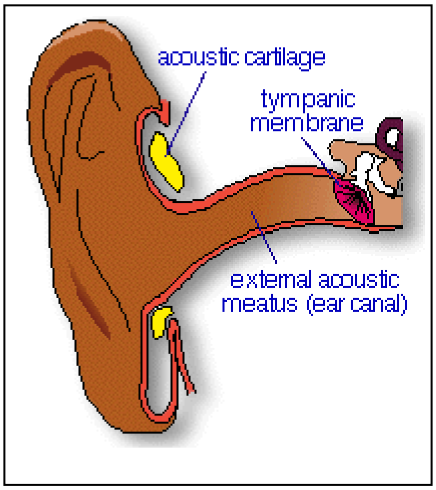
During speech, air from the lungs is pushed through the vocal cords (folds) then through the vocal tract where it is altered by the shape of the vocal tract before being emitted from the lips. Vocal cords are elastic protuberances of tendon, muscle and mucous membrane. Their tension, elasticity, length, width and separation are altered during speech.

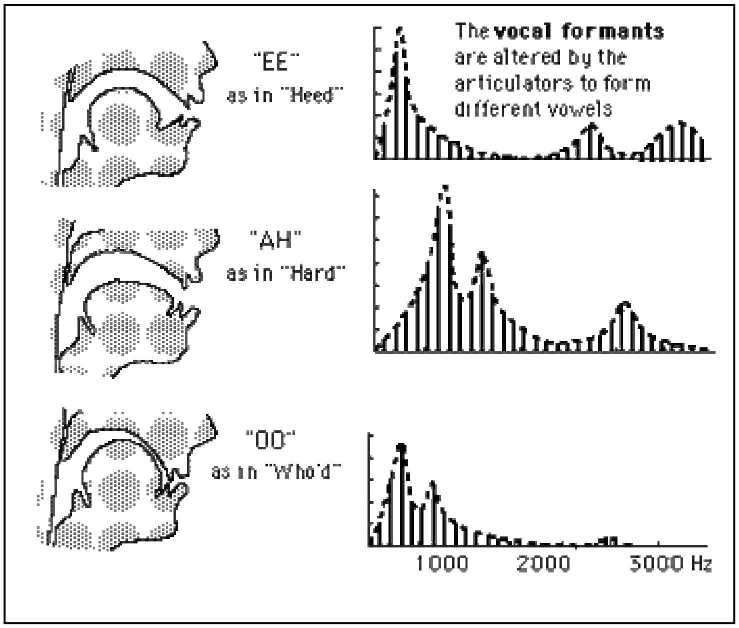
The output of the vocal cords has a fundamental frequency of about 150 Hz for men, 200 Hz for women and 300 Hz for children. These frequency differences are attributed to differences in vocal cord size (17-24mm long in men and 13-17mm long in women). This fundamental frequency is altered during speech. In addition to the fundamental frequency, the output of the vocal cords contains harmonics of the fundamental frequency which fall off in intensity at a rate of about 12 dB for each doubling of frequency (octave).



The vocal tract begins at the vocal cords and ends at the mouth. It is a sound resonator analogous to an organ pipe. The vocal tract approximates a tube closed at one end (vocal cords) and open at the other (lips). The fundamental resonant frequency will have a wavelength 4 times the length of the tube. A male vocal tract may be about 17 cm long. This corresponds to a fundamental frequency, $f = v/4L= (343m/s)/(4x0.17m) = 500Hz$, known as the formant F1. The higher harmonics follow $f_n = mv/4L$; i.e. F2 has f =1500 Hz and F3 has f = 2500 Hz. The resonances of the vocal tract modify the frequency spectrum of the vocal cords by enhancing vibrations near each of the resonances and suppressing off resonance vibrations. This results in three major peaks or formants in the frequency spectrum of speech. The frequency of each formant can be altered if the shape of the vocal tract is altered near a point of maximum velocity (displacement antinode) or a point of maximum pressure (displacement node). The vocal cords are always a point of maximum pressure and the lips are always a point of maximum velocity. Constrictions at point of maximum velocity lower the resonant frequency while constrictions at points of maximum pressure raise the resonance frequency. The first formant (F1) is most responsive to changes in mouth opening; small mouth openings lower the frequency of F1 and large openings raise the frequency of F1. The second formant F2 is most responsive to changes in the oral cavity; tongue backing or lip activity lower F2 while constrictions at the tongue would raise F2. F3 is responsive to front versus back constrictions. Positions of the mouth and the frequencies of the different formants have been characterized while producing different periodic (vowel) sounds.

Speech spectra for three sounds and the shape of the vocal tract while producing the sounds: 'Heed', 'Hard' and 'Who'd'





Hearing starts at the opening of the ears which face out and forward. Combining sound inputs from both ears enables the listener to determine direction from the relative phases of the sound. The next chamber, called the ear canal, has a dual role in hearing: 1) protection of the delicate inner parts of the ear and 2) it acts as a quarter wavelength resonator. This resonant cavity works such that the maximum air pressure (displacement node) is at the tympanic membrane at the inner end and the maximum air particle velocity (displacement antinode) is at the ear opening. In the average adult, the auditory canal is about 2.7 cm long. The fundamental frequency of the channel is given by $f_1 = v_s/4L = (343\text{m/s})/(4\times0.027\text{m}) = 3.2$ kHz. This is the frequency we hear best and it corresponds to about the centre of the speech frequency band (0.1 - 5kHz).

Section 21.8 Beats

Superposition of waves of different frequencies produces <u>temporal</u> interference

<u>Beating</u> is the periodic variation in intensity at a given point due to the superposition of two waves of slightly different frequencies.

Consider two sine waves of equal amplitude travelling through a medium with slightly different frequencies f_1 and f_2.

Lets suppose that we are sitting still so we needn't worry about the spatial dependence.

At x=0, the time dependence of these waves follows:

$$D_1 = D_M \sin\omega_1 t; \qquad D_2 = D_M \sin\omega_2 t \qquad \text{where } \omega_1 = 2\pi f_1 \text{ and } \omega_2 = 2\pi f_2$$

$$\sin(a) + \sin(b) = 2\cos[(a-b)/2]\sin[(a+b)/2]$$

$$\text{then } D = D_1 + D_2 = 2D_M\cos[(\omega_1-\omega_2)/2t]\,\sin[(\omega_1+\omega_2)/2t]$$

Or in terms of frequency:

$$\mathbf{D = 2D_M\cos[2\pi(f_1\text{-}f_2)/2t]\sin[2\pi(f_1+f_2)/2t]}$$

We treat the cosine term as an amplitude which varies at frequency $(f_1 - f_2)/2$ and the sine term as an oscillation at frequency $(f_1 + f_2)/2$. If this were a sound, the listener would hear the frequency $(f_1 + f_2)/2$ but its amplitude would vary up and down as $(f_1 - f_2)/2$. However, since there are two maxima, one positive and the other negative, which we cannot distinguish, the frequency of the 'beat' is $|f_1 - f_2|$.

Note that these beats occur as the two input sine waves come into phase and go out of phase with each other.

Our ears can only pick up beats with frequency less than 20 beats/s.

Problem:

While attempting to tune the note C at 523Hz, a piano tuner hears 2 beats/s between a reference oscillator (at 523 Hz) and the string.

(a) What are the possible frequencies of the piano wire?

Answer $y = y_{piano} + y_{523Hz} = 2A\cos[2\pi(f_{piano} - 523Hz)/2t]\ \sin[2\pi(f_{piano} + 523Hz)/2t]$

Then, $|f_{piano} - 523\ Hz| = 2Hz$, then $f_{piano} = 523Hz \pm 2Hz$ i.e. 521 or 525 Hz

(b) After tightening the string, she hears 3 beats/s, What is the string frequency now?

The beat frequency increased from 2 to 3 Hz.

Recall the $f_n = (m/2L)(\sqrt{T_s/\mu})$; so increasing tension increases the frequency. In fact,

$f_1/f_2 = \sqrt{T_{s1}}/\sqrt{T_{s2}}$ or $T_{s1}/T_{s2} = f_1^2/f_2^2$

Hence the frequency $f_{piano} = 523 + 3Hz = 526$ Hz

The solution $f_{piano} = 523\text{-}3=520Hz$ is not possible since that would correspond to a decrease in tension.

By what percentage should the piano tuner change the tension to bring the wire into tune.

$T_{s2}/T_{s1} = f_2^2/f_1^2 = 523^2/526^2 = 0.989$ or a 1.14% decrease in tension.