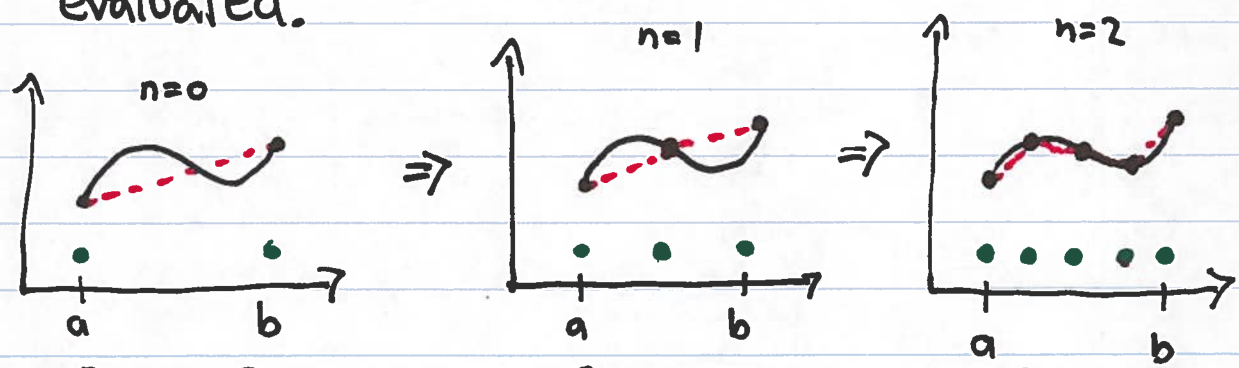
Computational Physics

Notes on Romberg Integration

Romberg integration applies Richardson extrapolation to a series of trapazoidal approximants with larger numbers of points (smaller step sizes). It fills in the points between those already evaluated.

Series of Trapazoidal Approximants



$2^n+1 = 2$ $\quad$ $2^n+1 = 3$ $\quad$ $2^n+1 = 5$

T_n^0 is the trapazoidal rule with 2^n+1 points

$$T_0^0 = \frac{1}{2}\left[f(a)+f(b)\right](b-a)$$

$$T_1^0 = \left[\frac{1}{4}f(a)+\frac{1}{2}f((a+b)/2)+\frac{1}{4}f(b)\right](b-a)$$

$$= \frac{1}{2}T_0^0 + \frac{1}{2}f\left(a+\frac{1}{2}(b-a)\right)(b-a)$$

$$h_n = \frac{b-a}{2^n}$$

stepsize

$$(*) \quad T_n^0 = \frac{1}{2}T_{n-1}^0 + h_n \sum_{j=1}^{2^{n-1}} f(a+(2k-1)h_n)$$

Notes on Romberg Integration cont.

So to implement Romberg integration first write a routine that can generate the Trapazoidal rule with 2^n+1 points T_n^0.

(This could be based on formula (*) or just some other N-point routine with $N = 2^n+1$)

We know that the trapazoidal rule has $O(h^2)$ error term so we can build a first Richardson extrapolation as

$$T_n^1 = \frac{1}{3}\left[4T_n^0 - T_{n-1}^0\right] = \frac{1}{2^2-1}\left[2^2 T_n^0 - T_{n-1}^0\right]$$

The m^{th} Richardson extrapolation, since T_n^{m-1} has $O(h^{2m})$ error term is

$$T_n^m = \frac{1}{4^m-1}\left[4^m T_n^{m-1} - T_{n-1}^{m-1}\right]$$

has errorterm $O(h^{2m+2})$

Notes on Romberg Integration cont.

So for a 5 times iterated Romberg Integration we have the dependencies

$$
\begin{array}{cccccc}
T_0^0 & & & & & \\
T_1^0 & T_1^1 & & & & \\
T_2^0 & T_2^1 & T_2^2 & & & \\
T_3^0 & T_3^1 & T_3^2 & T_3^3 & & \\
T_4^0 & T_4^1 & T_4^2 & T_4^3 & T_4^4 & \\
T_5^0 & T_5^1 & T_5^2 & T_5^3 & T_5^4 & T_5^5
\end{array}
$$

If you calculate

$T_0^0, T_1^0, T_2^0, T_3^0, T_4^0, T_5^0$

you can find T_5^5 by repeated Richardson extrapolation

With $T_0^0, \sim\sim, T_n^0$ you can find T_n^n

Notes on Romberg Integration cont.

E.g. $\int_1^2 \frac{1}{x}\,dx = \ln 2$

By hand ~

$$T_0^0 = \frac{1}{2}\left[1 + \frac{1}{2}\right] = 0.75$$
$$T_1^0 = \frac{1}{2}T_0^0 + \frac{1}{2}\left(\frac{1}{1.5}\right) = 0.708333333$$
$$T_2^0 = 0.69702380952$$
$$T_3^0 = 0.69412185037$$
$$T_4^0 = 0.69339120220$$

So

$$\begin{array}{lllll} T_0^0 & & & & \\ T_1^0 & T_1^1 & & & \\ T_2^0 & T_2^1 & T_2^2 & & \\ T_3^0 & T_3^1 & T_3^2 & T_3^3 & \\ T_4^0 & T_4^1 & T_4^2 & T_4^3 & T_4^4 \end{array}$$

||

0.75000000000				
0.70833333333	0.69444444444			
0.69702380952	0.69325396825	0.69317460317		
0.69412185037	0.69315453065	0.69314790148	0.69314747764	
0.69339120220	0.69314765281	0.69314719429	0.69314718307	0.69314718191

The correct digits are shown in bold (the exact answer to 15 digits is given by $\ln 2 = 0.693147180559945$)