Challenge problem #3.

(b) $$S(y+\alpha\eta) = \int_{x_0}^{x_1} dx\, f(y+\alpha\eta,\, y'+\alpha\eta',\, x)$$

$$\left.\frac{dS}{d\alpha}\right|_{\alpha=0} = \int_{x_0}^{x_1} dx \cdot \left(\frac{\partial f}{\partial y}\cdot\eta + \frac{\partial f}{\partial y'}\eta'\right)$$

$$= \int_{x_0}^{x_1} dx\, \frac{\partial f}{\partial y}\eta + \int_{x_0}^{x_1} \frac{\partial f}{\partial y'}\, d\eta$$

$$= \int_{x_0}^{x_1} dx\, \frac{\partial f}{\partial y}\eta + \eta\frac{\partial f}{\partial y'}\Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta\, \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) . dx$$

$$= \int_{x_0}^{x_1} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

(a) $$f = \sqrt{1+y'^2}$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 - \left(\frac{y'}{\sqrt{1+y'^2}}\right)' = 0 \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = C \Rightarrow y'^2 = \frac{C^2}{1-C^2}$$

$$\Rightarrow \quad \cancel{y''=0 \text{ or } y'=0} \quad y' = a = \pm\sqrt{\frac{C^2}{1-C^2}}$$

$$\Rightarrow \quad y = ax + b,$$ where a, b are constants.