Conservation of Momentum

One consequence of Newton's third law is the conservation of momentum for a multiparticle system.

Two Particles:

Consider 2 particles that may exert forces (gravitational, electromagnetic, etc...) on each other and also be subject to external forces.

2nd Law for #1 $\quad \vec{F}_1 = \vec{F}_{12} + \vec{F}_1^{ext} = \dot{\vec{p}}_1$

2nd Law for #2 $\quad \vec{F}_2 = \vec{F}_{21} + \vec{F}_2^{ext} = \dot{\vec{p}}_2$

If we define the total momentum

$$\vec{P} = \vec{p}_1 + \vec{p}_2$$

Conservation of Momentum con't.

Then, we find (adding the two equations)

$$\dot{\vec{P}} = (\vec{F}_{12} + \vec{F}_{21}) + (\vec{F}_1^{\,ext} + \vec{F}_2^{\,ext})$$

Let's call $\vec{F}^{ext} = \vec{F}_1^{\,ext} + \vec{F}_2^{\,ext}$ the total force on this two-particle system.

By the 3rd Law $\vec{F}_{21} = -\vec{F}_{12}$

$$\Rightarrow \qquad \dot{\vec{P}} = \vec{F}^{ext}$$

$\therefore$ If $\vec{F}^{ext} = 0$ (no external total force)

$$\vec{P} = \vec{p}_1 + \vec{p}_2 = \text{constant}$$

Conservation of Momentum cont.

N Particles:

2^{nd} Law for the α^{th}

$$\forall\alpha \qquad \vec{F}_\alpha = \sum_{\beta\neq\alpha} \vec{F}_{\alpha\beta} + \vec{F}_\alpha^{ext} = \dot{p}_\alpha$$

Total Momentum: $\vec{P} = \sum_\alpha \vec{p}_\alpha$

$$\Rightarrow \quad \dot{\vec{P}} = \sum_\alpha \dot{\vec{p}}_\alpha$$

$$\dot{\vec{P}} = \sum_\alpha \sum_{\beta\neq\alpha} \vec{F}_{\alpha\beta} + \sum_\alpha \vec{F}_\alpha^{ext}$$

but $\sum_\alpha \sum_{\beta\neq\alpha} \vec{F}_{\alpha\beta} = \sum_\alpha \sum_{\beta>\alpha} (\vec{F}_{\alpha\beta} + \vec{F}_{\beta\alpha}) = 0$ 3^{rd} Law

$$\Rightarrow \quad \dot{\vec{P}} = \sum_\alpha \vec{F}_\alpha^{ext}$$

$$\therefore \quad \vec{P} = \sum_\alpha \vec{p}_\alpha = \text{constant}$$

if no net external forces

Conservation of Momentum

For a system of N particles $\alpha = 1, \ldots, N$

Total Momentum $\vec{P} = \vec{p}_1 + \cdots + \vec{p}_N = \sum \vec{p}_\alpha$

Newton's 2^{nd} + 3^{rd} Laws $\Rightarrow$ $\dot{\vec{P}} = \vec{F}^{ext}$

Net External Force on the system

Principle of Conservation of Momentum

If the net external force $\vec{F}^{ext}$ on a N-particle system is zero, the system's total mechanical momentum $\vec{P} = \sum m_\alpha \vec{v}_\alpha$ is constant.

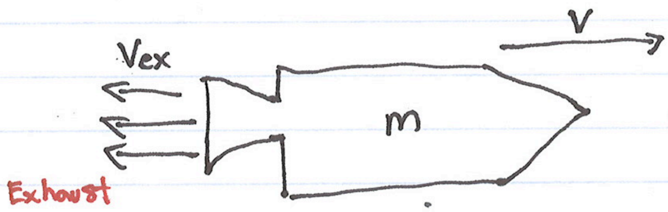
We will see later that this principle follows from the translation invariance of the system.

Physics is the same if we shift the origin!

Constancy of Momentum of system of N particles iff Newton's 3^{rd} Law

"Conservation" of Momentum is more general but must consider objects outside of system in general.

Rocket Propulsion



If $P(t)$ is the total momentum of the rocket and fuel at time t then

$$P(t+dt) = \underbrace{(m+dm)(V+dv)}_{\text{Rocket}} - \underbrace{dm\,(V-V_{ex})}_{\text{Exhaust}}$$

$$P(t+dt) = mv + mdv + V_{ex}dm$$

$$\Rightarrow \quad dP = mdv + V_{ex}dm$$

Generally, $\quad \dot{P} = m\dot{v} + V_{ex}\dot{m}$

If isolated $\quad m\dot{v} = \underbrace{-\dot{m}V_{ex}}_{\text{Thrust}}$

$$dV = -V_{ex}\frac{dm}{m}$$

$$\Rightarrow \quad V(t) = V_0 + V_{ex}\ln\left(\frac{m_0}{m(t)}\right)$$