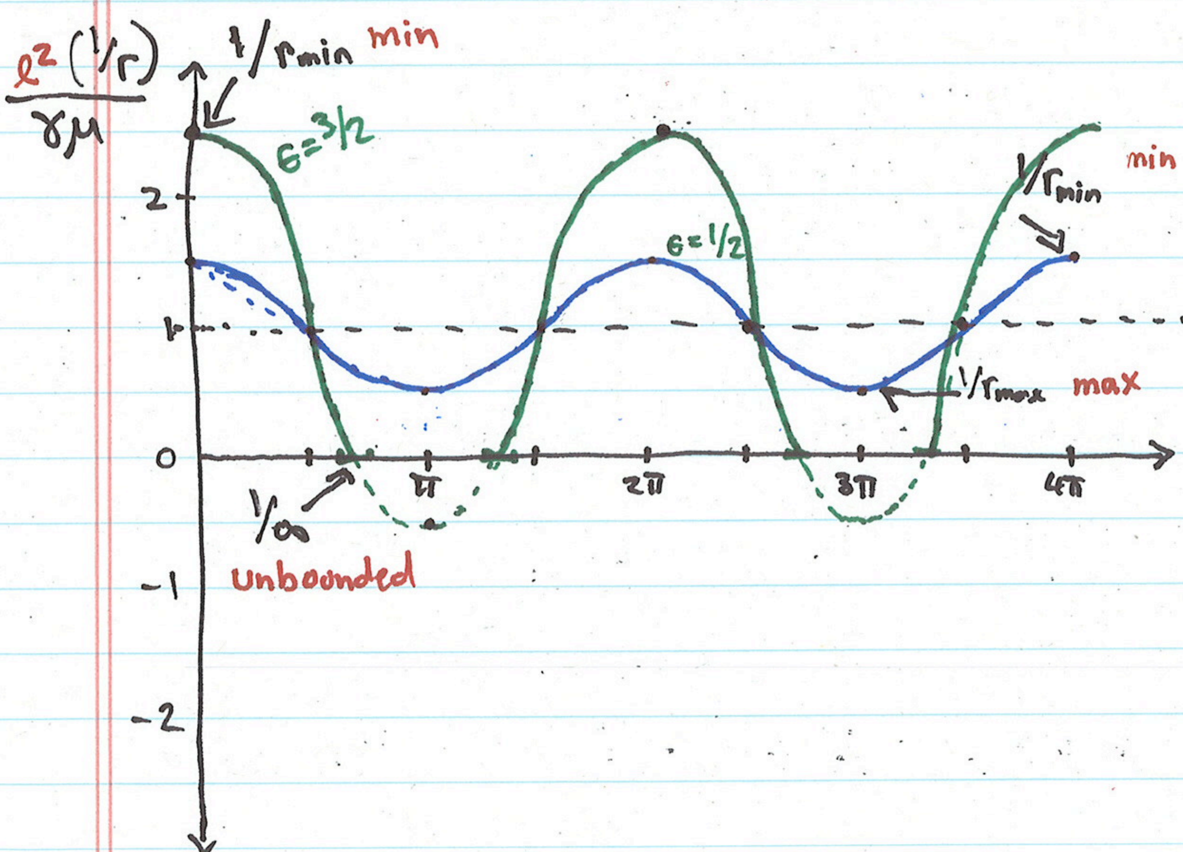
Kepler Problem

So $u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos\phi$

eccentricity

or $u(\phi) = \frac{\gamma\mu}{\ell^2}\left(1 + \epsilon\cos\phi\right)$

$\therefore \quad r(\phi) = \frac{(\ell^2/\gamma\mu)}{1 + \epsilon\cos\phi} = \frac{c}{1+\epsilon\cos\phi}$



Kepler Problem

Bounded Orbits $\quad 0 \leq \epsilon < 1$

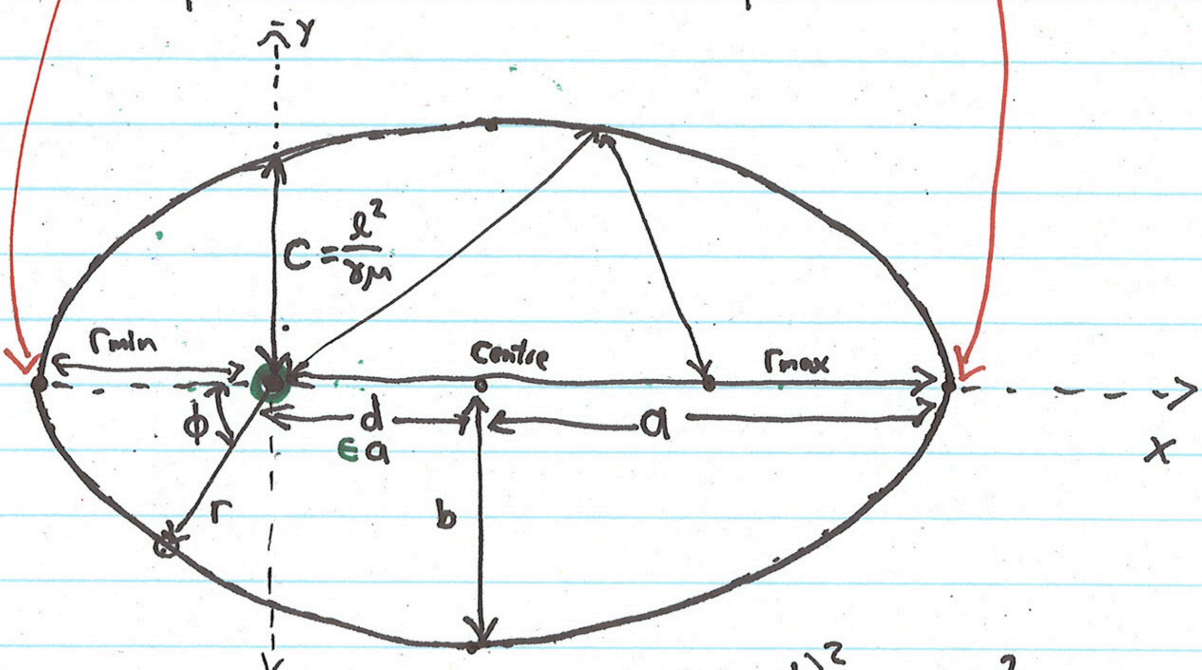
In this case the particle/planet oscillates between

Note: Circle if $\epsilon = 0$

$$r_{min} = \frac{c}{1+\epsilon} \quad \text{and} \quad r_{max} = \frac{c}{1-\epsilon}$$

perihelion $\qquad\qquad$ aphelion

$\phi = 0$ $\qquad\qquad$ $\phi = \pi$



$$r(\phi) = \frac{c}{1+\epsilon\cos\phi} \iff \frac{(x-d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

ellipse with FOCUS at origin

Kepler Problem

Bounded Orbits

Geometry gives the relationships:

semimajor axis   semiminor axis

$$a = \frac{C}{1-\epsilon^2} \qquad b = \frac{C}{\sqrt{1-\epsilon^2}} \qquad d = a\epsilon$$

$$b/a = \sqrt{1-\epsilon^2}$$

ϵ eccentricity

$$C = \frac{\ell^2}{\gamma\mu}$$

determined by $|\vec{\ell}| = \ell$ magnitude of angular momentum only if γ + μ fixed

By the way....

This is Kepler's First Law:

Planets (and other bound heavenly bodies) follow orbits that are ellipses with the Sun at one focus.

Kepler Problem

Recall that Kepler's Second Law states:

$$\frac{dA}{dt} = \frac{\ell}{2\mu}$$

"Equal areas in equal times"

For an ellipse $A = \pi ab$, from which we can deduce that, as the total area is swept out in a period τ

$$\tau = \frac{A}{dA/dt} = \frac{2\pi ab\mu}{\ell}$$

$$\tau^2 = 4\pi^2 \frac{a^2b^2\mu^2}{\ell^2} = \frac{4\pi^2 a^4(1-\epsilon^2)\mu^2}{\ell^2} = \frac{4\pi^2 a^3 c\mu^2}{\ell^2}$$

$$\tau^2 = 4\pi^2 \frac{a^3\mu}{\gamma}$$

For the sun $\gamma = G m_1 m_2 \simeq G\mu M_s$

$$\tau^2 = \frac{4\pi^2}{GM_s} a^3$$

Kepler's Third Law: For all bodies orbiting the Sun the square of the period is proportional to the cube of the semi-major axis

Kepler Problem

Energy in the Kepler Problem

$$E = T + U = \frac{1}{2}m\dot{r}^2 + U_{eff}$$

at a turning point $\dot{r}=0$ $\quad E = U_{eff}(r_{min})$

$$E = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2}$$

$$E = \frac{1}{2r_{min}}\left(\frac{\ell^2}{\mu r_{min}} - 2\gamma\right) \qquad r_{min} = \frac{c}{1+\epsilon} = \frac{\ell^2}{\gamma\mu(1+\epsilon)}$$

$$E = \frac{\gamma\mu(1+\epsilon)}{2\ell^2}\left[\gamma(1+\epsilon) - 2\gamma\right]$$

$$\therefore \quad E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1) = \frac{\gamma}{2c}(\epsilon^2-1)$$

$\epsilon < 1$	$\epsilon \geq 1$
$E < 0$	$E \geq 0$
bound	unbound

Written in terms of $a = \frac{c}{1-\epsilon^2} = -\alpha$

$$E = \frac{-\gamma}{2a} = \frac{\gamma}{2\alpha}$$

$a = -\alpha$ determined only by energy E

Kepler Problem

Unbound Orbits $\quad \epsilon \geq 1$

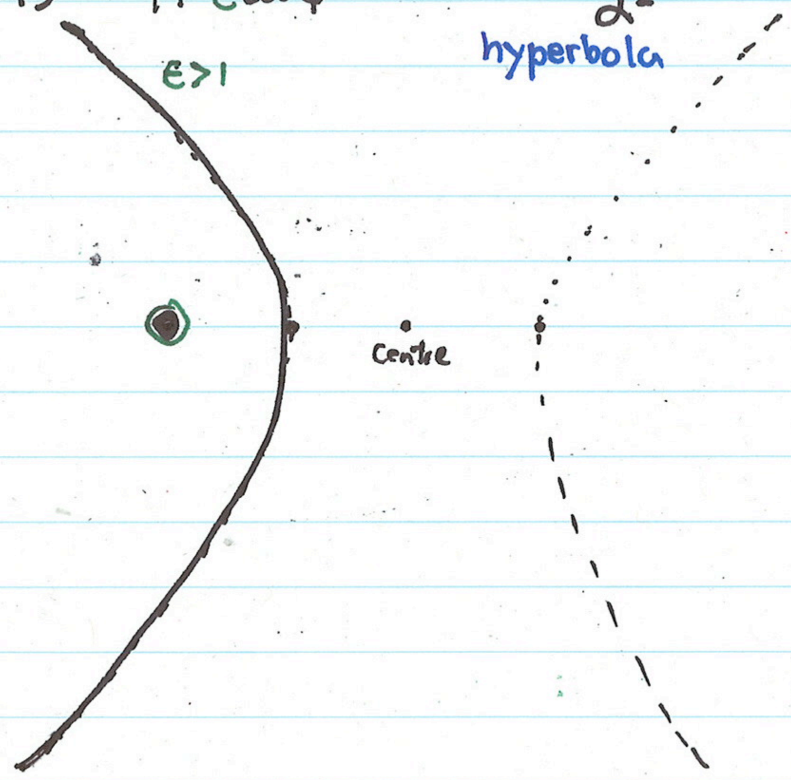
In this case the particle/object has a minumum at

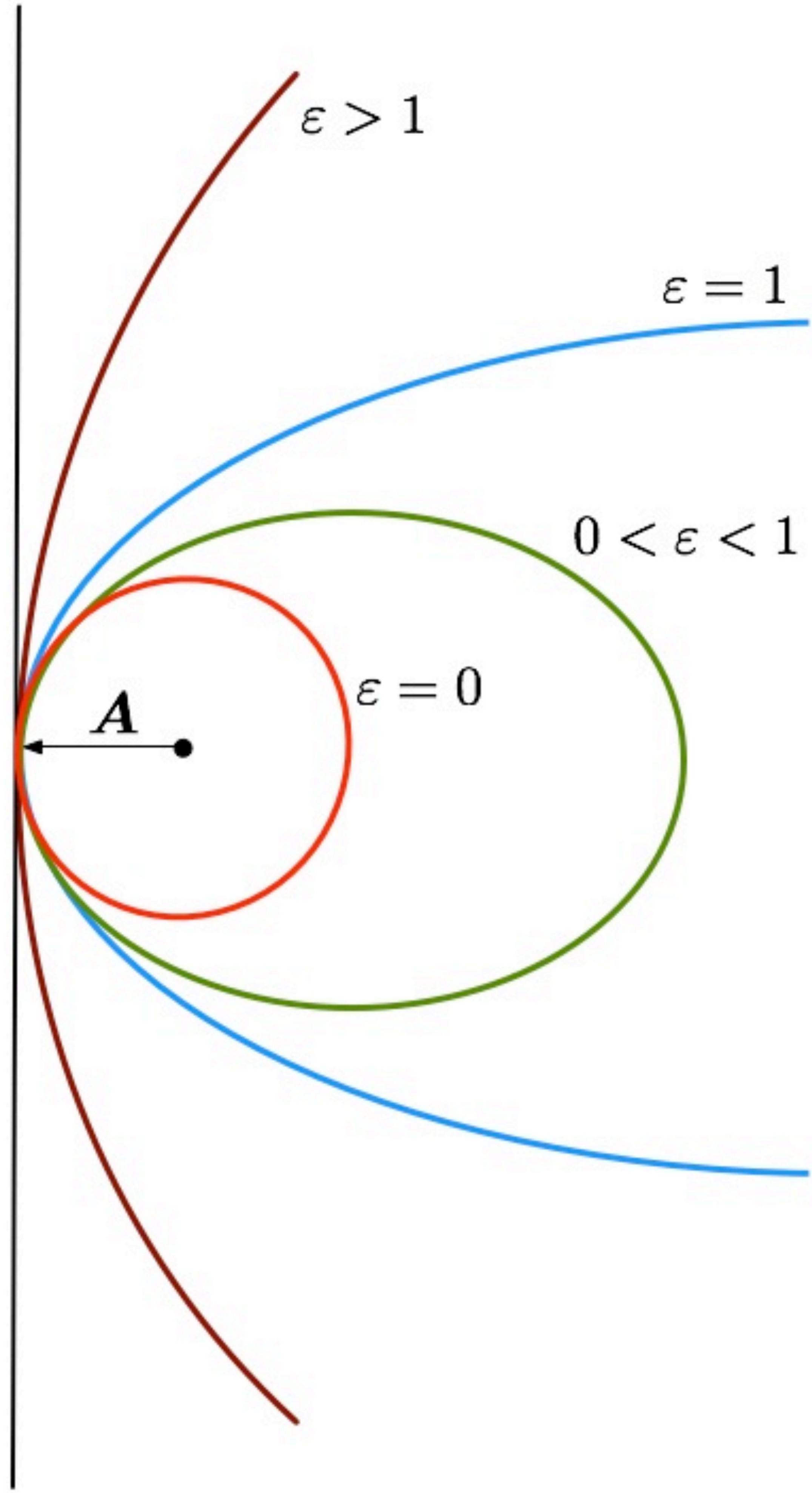
$$r_{min} = \frac{c}{1+\epsilon} \quad \text{and} \quad \text{no } r_{max}$$

Since $\epsilon \geq 1$ $\quad E \geq 0$
and the object may "escape" to $r \to \infty$
with $E = \lim_{r\to\infty} (T+U) = T_\infty \geq 0$

$$r(\phi) = \frac{c}{1+\epsilon\cos\phi} \iff \frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$$

hyperbola

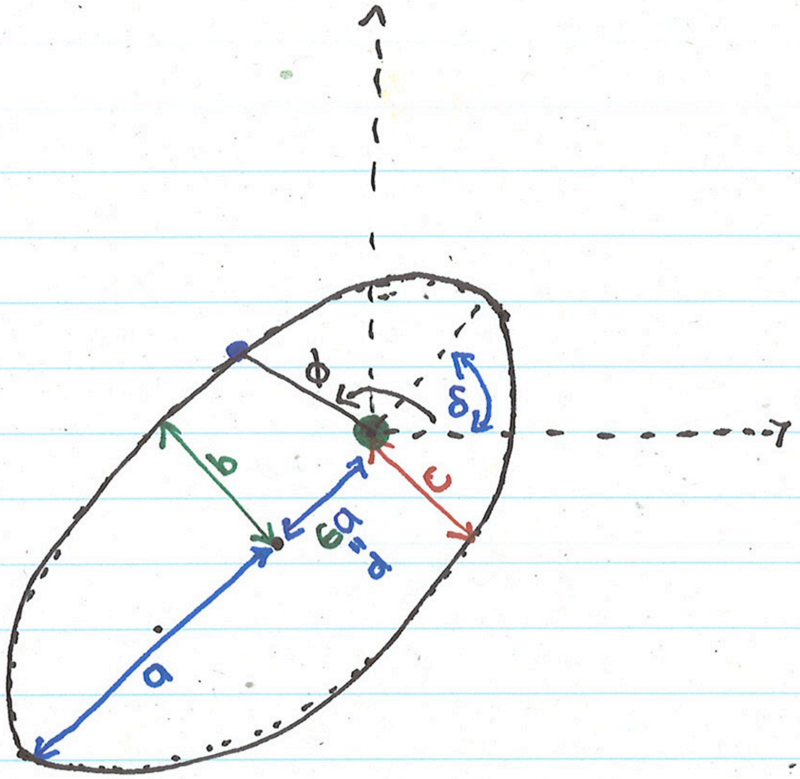




$\varepsilon > 1$
$\varepsilon = 1$
$0 < \varepsilon < 1$
$\varepsilon = 0$
$\boldsymbol{A}$

Kepler Problem

Orbital Parameters



$$U = -\frac{\gamma}{r}$$

$$r(\phi) = \frac{c}{1+\epsilon\cos(\phi-\delta)}$$

Hyperbola $\epsilon \geq 1$

$$\alpha = (\epsilon^2 - 1)\, c$$

Ellipse $\epsilon < 1$

$$\begin{cases} c = (1-\epsilon^2)a = \sqrt{1-\epsilon^2}\, b \\ b/a = \sqrt{1-\epsilon^2} \qquad d = \epsilon a \end{cases}$$

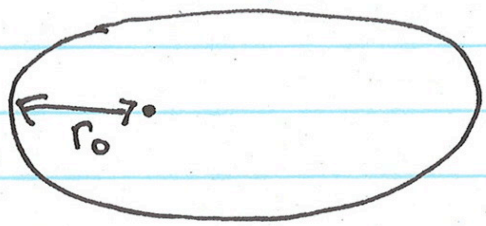
Angular Momentum $\quad c = \frac{\ell^2}{\gamma\mu} \qquad \ell = \sqrt{\gamma\mu c} = |\vec{r} \times \vec{p}|$

Energy $\quad E = -\frac{\gamma}{2a} = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1) = T + U$

$$= \tfrac{1}{2}\mu\dot{r}^2 + U_{eff}$$

Kepler Problem

Example: Spaceship I



$\epsilon = \frac{1}{2}$

a) What is r_{max} (in terms of r_0)?

b) What are E and ℓ (in terms of r_0)?