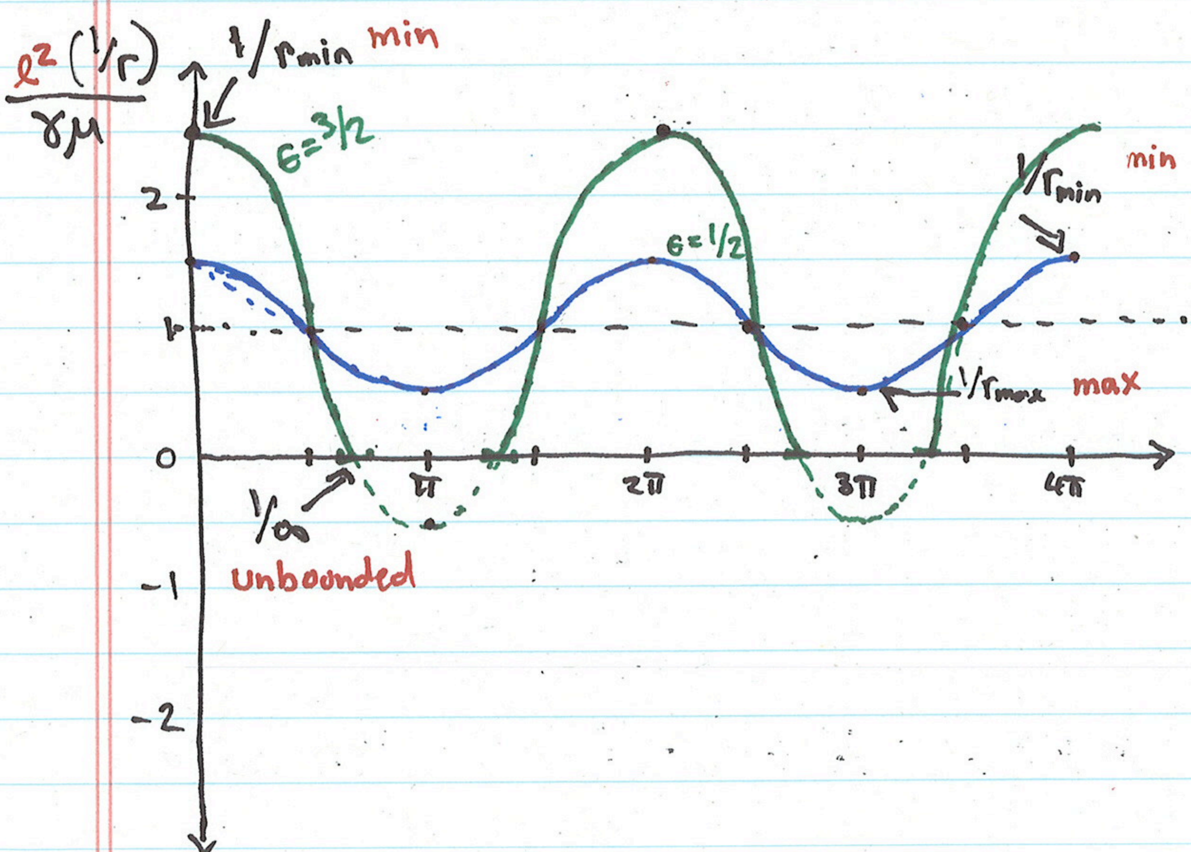
Kepler Problem

So $u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos\phi$

eccentricity

or $u(\phi) = \frac{\gamma\mu}{\ell^2}\left(1 + \epsilon\cos\phi\right)$

$\therefore \quad r(\phi) = \frac{(\ell^2/\gamma\mu)}{1+\epsilon\cos\phi} = \frac{c}{1+\epsilon\cos\phi}$



Kepler Problem

Bounded Orbits $\quad 0 \leq \epsilon < 1$

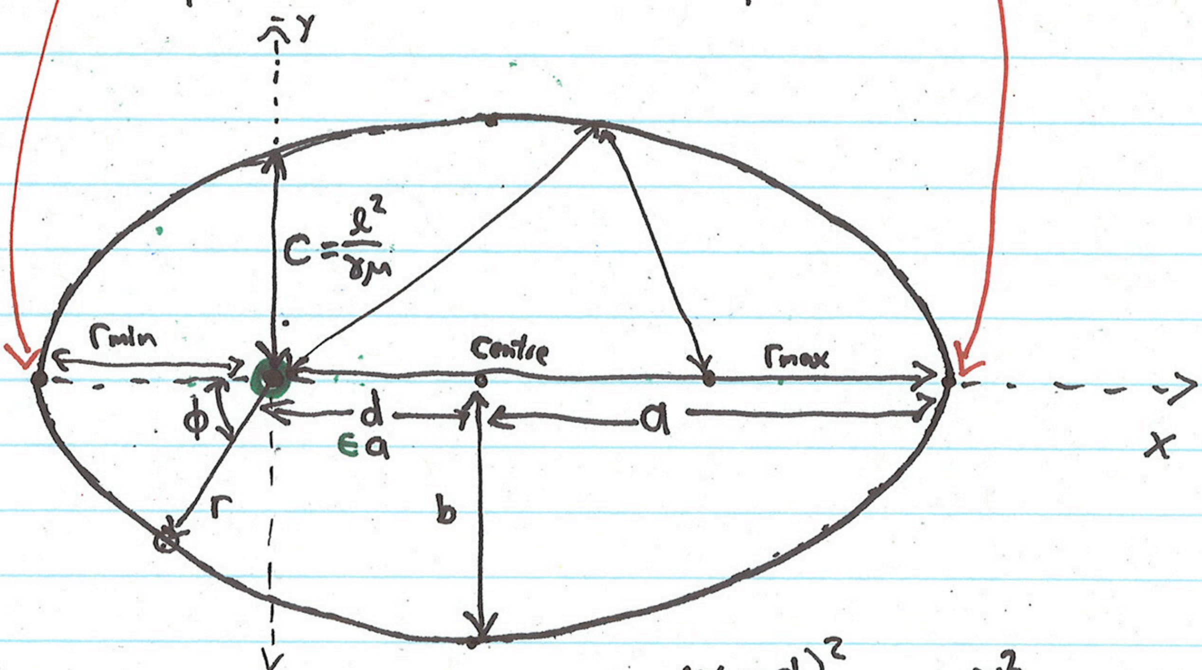
In this case the particle/planet oscillates between

Note: Circle if $\epsilon = 0$

$$r_{min} = \frac{c}{1+\epsilon} \quad \text{and} \quad r_{max} = \frac{c}{1-\epsilon}$$

perihelion $\quad$ aphelion

$\phi = 0 \quad \phi = \pi$



$$r(\phi) = \frac{c}{1+\epsilon\cos\phi} \iff \frac{(x-d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

ellipse with FOCUS at origin

Kepler Problem

Bounded Orbits

Geometry gives the relationships:

semimajor axis $\qquad$ semiminor axis

$$a = \frac{C}{1-\epsilon^2} \qquad b = \frac{C}{\sqrt{1-\epsilon^2}} \qquad d = a\epsilon$$

$$b/a = \sqrt{1-\epsilon^2}$$

ϵ eccentricity

$$C = \frac{\ell^2}{\gamma\mu}$$

determined by $|\vec{\ell}| = \ell$

magnitude of angular momentum

only if γ + μ fixed

By the way....

This is Kepler's First Law:

Planets (and other bound heavenly bodies) follow orbits that are ellipses with the Sun at one focus.

Kepler Problem

Recall that Kepler's Second Law states:

$$\frac{dA}{dt} = \frac{\ell}{2\mu}$$

"Equal areas in equal times"

For an ellipse $A = \pi ab$, from which we can deduce that, as the total area is swept out in a period τ

$$\tau = \frac{A}{dA/dt} = \frac{2\pi ab\mu}{\ell}$$

$$\tau^2 = 4\pi^2 \frac{a^2b^2\mu^2}{\ell^2} = \frac{4\pi^2 a^4(1-\epsilon^2)\mu^2}{\ell^2} = \frac{4\pi^2 a^3 c\mu^2}{\ell^2}$$

$$\tau^2 = 4\pi^2 \frac{a^3\mu}{\gamma}$$

For the sun $\gamma = Gm_1m_2 \simeq G\mu M_s$

$$\tau^2 = \frac{4\pi^2}{GM_s} a^3$$

Kepler's Third Law: For all bodies orbiting the Sun the square of the period is proportional to the cube of the semi-major axis