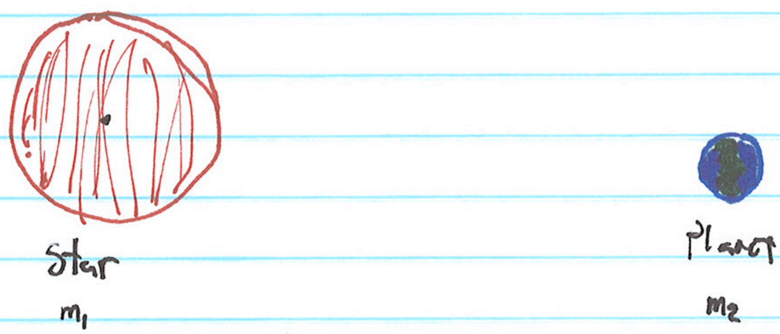
Lagrangian Mechanics

Kepler Problem

Not to Scale!



a) What is $U(r)$? $U(r) = -\frac{Gm_1m_2}{r} = -\frac{\gamma}{r}$

b) Lagrange equations

c) $U_{eff}(r) \rightarrow$ sketch Equivalent 1D problem

d) What kind of motion allowed for fixed E?

$$\mu\ddot{r} = -\frac{\partial U_{eff}}{\partial r}$$

Lagrangian Mechanics

Kepler Problem

$$U(r) = -\frac{Gm_1m_2}{r} = -\frac{\gamma}{r}$$

So $$L = T - U$$

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{\gamma}{r}$$

ϕ: $$\ell = \mu r^2\dot{\phi}$$

r: $$\mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{\gamma}{r^2}$$

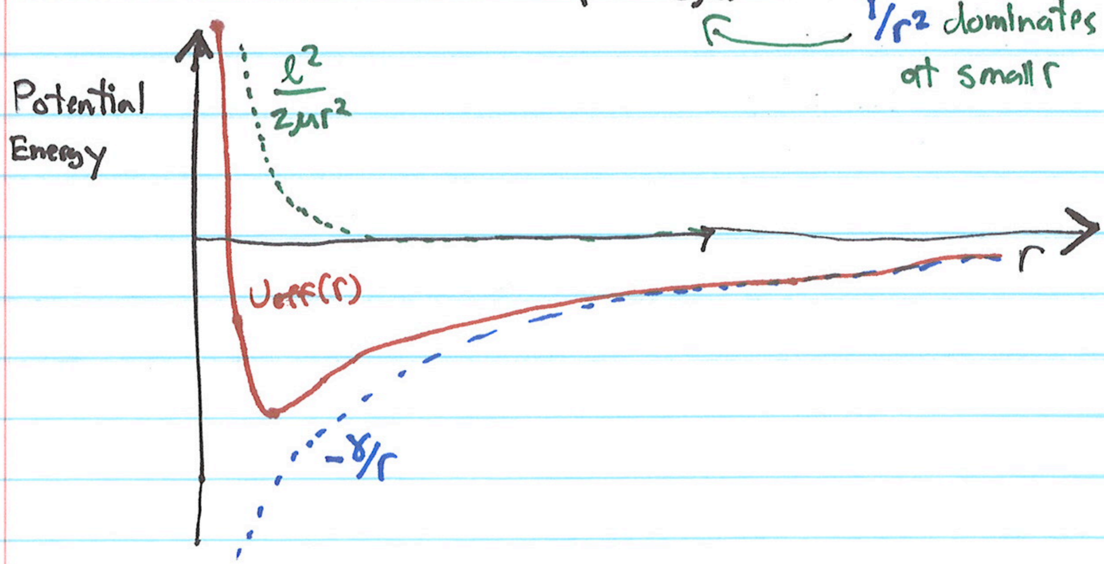
$$\frac{\partial L}{\partial r} = \mu r\dot{\phi}^2 - \frac{\gamma}{r^2} = \mu r\left(\frac{\ell}{\mu r^2}\right)^2 - \frac{\gamma}{r^2}$$

Effective Potential

$1/r$ dominates at larger r

$$U_{eff}(r) = -\frac{\gamma}{r} + \frac{\ell^2}{2\mu r^2}$$

$1/r^2$ dominates at small r



Kepler Problem

For a potential $U(r) = -\gamma/r$

The effective potential at fixed angular momentum ℓ is:

$$U_{eff}(r) = \frac{\ell^2}{2\mu r^2} + U(r)$$

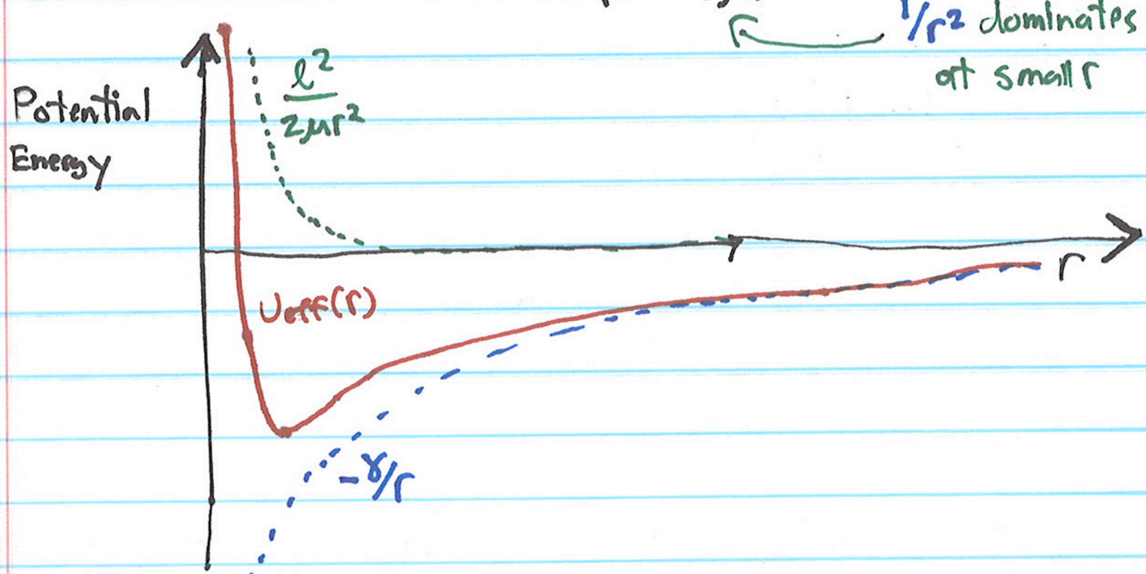
$$U_{eff}(r) = \frac{\ell^2}{2\mu r^2} - \frac{\gamma}{r}$$

Effective Potential

$$U_{eff}(r) = -\frac{\gamma}{r} + \frac{\ell^2}{2\mu r^2}$$

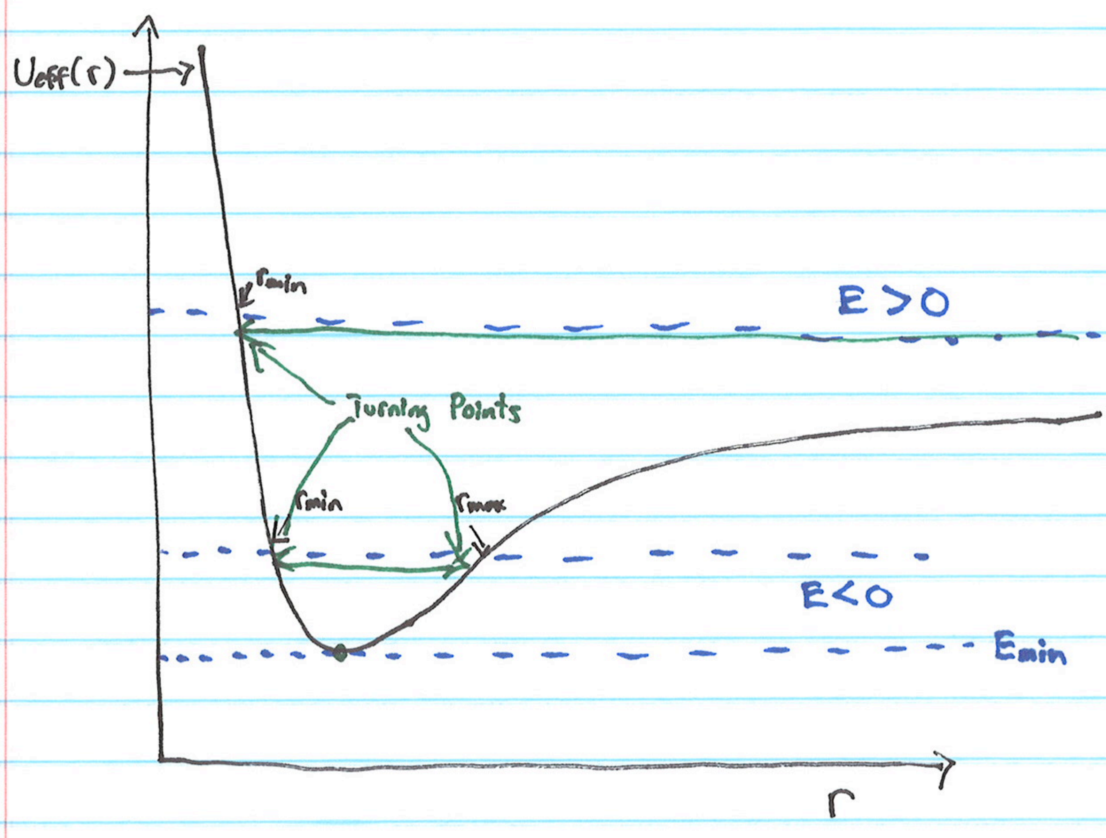
$1/r$ dominates at larger r

$1/r^2$ dominates at small r



Lagrangian Mechanics

Kepler Problem



Unbound $\infty > r > r_{min}$

Bound $r_{max} \geq r \geq r_{min}$

Solve $E = U_{eff}(r_{min})$ or $E = U_{eff}(r_{max})$
to find r_{min}, r_{max}

Kepler Problem

The radial equation is

Gravity

$$\mu \frac{d^2 r}{dt^2} = \mu \ddot{r} = -\frac{\partial U_{eff}}{\partial r} = -\frac{\gamma}{r^2} + \frac{\ell^2}{2\mu r^3}$$

Angular Momentum Conservation

sove for $r(t)$

We can use the fact that $\ell = r^2 \frac{d\phi}{dt} \mu$ to write $dt = \frac{r^2 \mu}{\ell} d\phi$

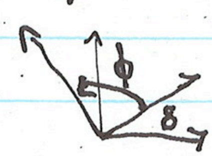
$$\Rightarrow \quad \frac{d}{dt} = \frac{\ell}{r^2 \mu} \frac{d}{d\phi} = \frac{\ell}{\mu} u^2 \frac{d}{d\phi}$$

If we define $u = 1/r$ then the radial equation becomes

$$\mu \frac{\ell}{\mu} u^2 \frac{d}{d\phi}\left(\frac{\ell}{\mu} u^2 \frac{d}{d\phi}\left(1/u\right)\right) = -\gamma u^2 + \frac{\ell^2}{2\mu} u^3$$

$$\Rightarrow \quad \frac{d^2 u}{d\phi^2} + u = \frac{\gamma \mu}{\ell^2}$$

"Shifted Harmonic Oscillator Equation"



$$\Rightarrow \quad u(\phi) = \frac{\gamma \mu}{\ell^2} + A\cos(\phi + \delta)$$

We can always set $\delta = 0$ by redefining $\phi = 0$

Lagrangian Mechanics

Kepler Problem: The Orbit

$$\frac{d^2 u}{d\phi^2} + u(\phi) = \frac{\gamma\mu}{\ell^2}$$

$$\Rightarrow \frac{d^2}{d\phi^2}\left(u - \frac{\gamma\mu}{\ell^2}\right) + \left(u - \frac{\gamma\mu}{\ell^2}\right) = 0$$

$$\Rightarrow \quad u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos(\phi + \delta)$$

Can choose $\delta = 0$ by redefining $\phi = 0$

$$\therefore \quad u(\phi) = \frac{\gamma\mu}{\ell^2}\left(1 + \epsilon\cos\phi\right)$$

$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi}$$

$$c = \frac{\ell^2}{\gamma\mu}$$

$\epsilon < 1$	ellipse
$\epsilon = 1$	parabola
$\epsilon \geq 1$	hyperbola

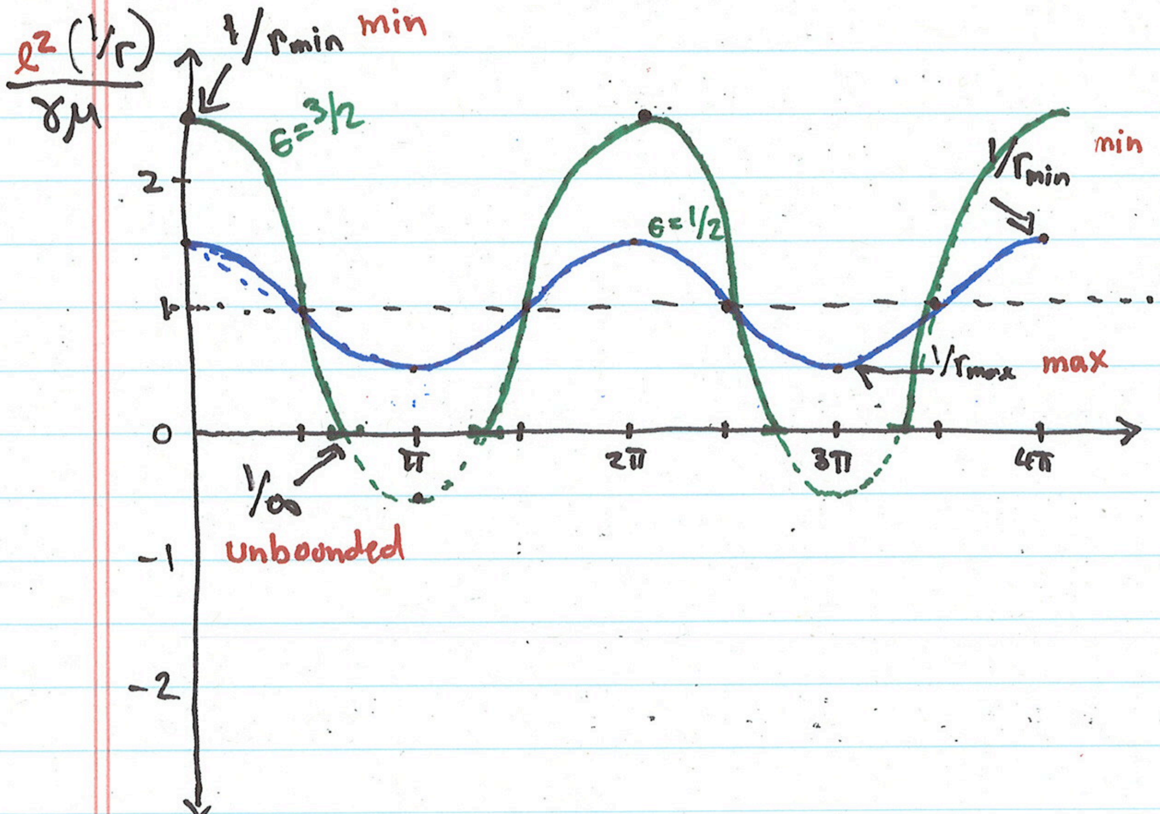
Kepler Problem

So $u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos\phi$

eccentricity

or $u(\phi) = \frac{\gamma\mu}{\ell^2}\left(1 + \epsilon\cos\phi\right)$

$$\therefore \quad r(\phi) = \frac{(\ell^2/\gamma\mu)}{1 + \epsilon\cos\phi} = \frac{c}{1+\epsilon\cos\phi}$$



Kepler Problem

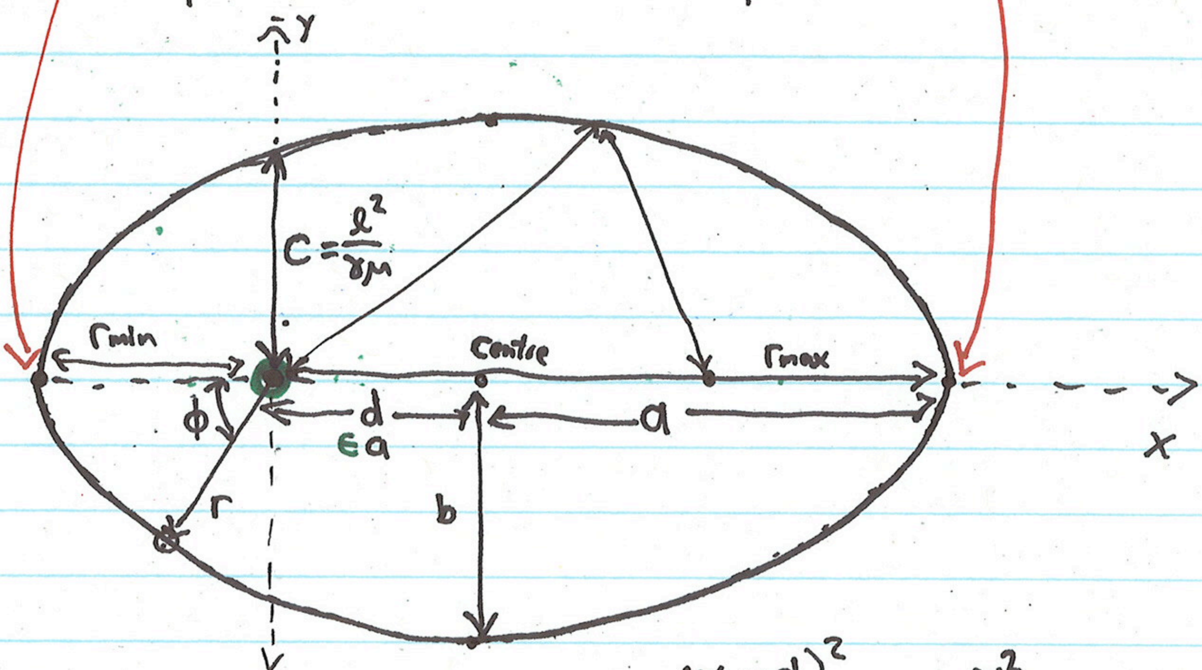
Bounded Orbits $0 \leq \epsilon < 1$

In this case the particle/planet oscillates between

Note: Circle if $\epsilon = 0$

$$r_{min} = \frac{c}{1+\epsilon} \quad \text{and} \quad r_{max} = \frac{c}{1-\epsilon}$$

perihelion $\phi = 0$ aphelion $\phi = \pi$



$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi} \iff \frac{(x-d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

ellipse with FOCUS at origin