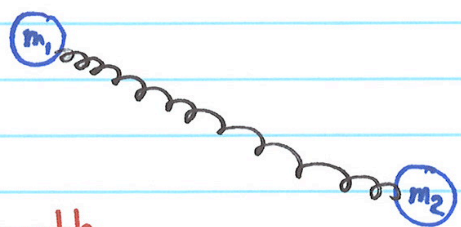
Lagrangian Mechanics

"Springs" in Free Space

Free to move and rotate in 3D



Assume $\ell = 0$
zero equilibrium length

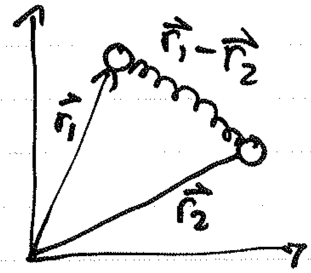
a) How many degrees of freedom?

b) Write down T and U in terms of $\vec{r}_1$, $\vec{r}_2$.

c) Choose generalized coordinates

d) Write down Lagrangian

e) Equations of Motion



a) Naively... 6

b) $\dot{\vec{r}}_1, \dot{\vec{r}}_2 \qquad \vec{r}_1, \vec{r}_2$

$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2}$$

$$U = \frac{1}{2} K (|\vec{r}_1 - \vec{r}_2|)^2 = \frac{1}{2} K (\vec{r}_1 - \vec{r}_2)^2$$

c) Choose $\vec{r}_1 - \vec{r}_2 = \vec{r}$

$$\vec{R} = \frac{m_1}{m_1+m_2} \vec{r}_1 + \frac{m_2}{m_1+m_2} \vec{r}_2$$

Springs in Free Space

In vector notation we have

$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2}$$ Kinetic Energy

$$U = \frac{1}{2} k (\vec{r}_1 - \vec{r}_2)^2$$ Potential Energy

Let's rewrite in terms of

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

then $U = U(|\vec{r}|) = U(r)$

choose $\vec{r}$ as one set of coordinates

and $\vec{R} = \frac{m_1 \vec{r}_1}{m_1 + m_2} + \frac{m_2 \vec{r}_2}{m_1 + m_2}$

or if $M = m_1 + m_2$

$$\left\{ \begin{aligned} \vec{r} &= \vec{r}_1 - \vec{r}_2 \\ \vec{R} &= \frac{m_1}{M} \vec{r}_1 + \frac{m_2}{M} \vec{r}_2 \end{aligned} \right.$$

Springs in Free Space

Rewrite KE: $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \quad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$

$$\Rightarrow \dot{\vec{r}}_1 = \dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}} \quad \dot{\vec{r}}_2 = \dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}$$

$$\Rightarrow T = \frac{1}{2}m_1\left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)^2$$

$$T = \frac{1}{2}\Big\{ (m_1+m_2)\dot{\vec{R}}^2 + \cancel{\frac{m_1 m_2}{M^2}(m_2\dot{\vec{r}} - m_1\dot{\vec{r}})\cdot\dot{\vec{R}}} \quad 0$$
$$+ \frac{m_1 m_2}{M}\dot{\vec{r}}^2 \Big\}$$

$$\Rightarrow \boxed{T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2}$$

$$\mu = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1+m_2}$$ reduced mass

$$\Rightarrow \quad L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - \frac{1}{2}k\vec{r}^2$$

Note that $\frac{\partial L}{\partial \vec{R}} = 0 = \frac{\partial L}{\partial X} = \frac{\partial L}{\partial Y} = \frac{\partial L}{\partial Z}$

These are ignorable coordinates. Why?

Springs in Free Space

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Z}}\right)$$

$$\Rightarrow \quad M\dot{X}\,\hat{i} + M\dot{Y}\,\hat{j} + M\dot{Z}\,\hat{k} = \vec{P} = \overrightarrow{\text{constant}}$$

Just tell us the total momentum conserved, (system as a whole is translation invariant)

$\vec{r}$ equations: $\quad L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - \frac{1}{2}k\vec{r}^2$

We have choosen to use $\vec{r}$ as a coordinate (actually 3) to characterize the system. But we have not chosen a representation of $\vec{r}$.

Actually... if we Keep cartesians in mind it is possible to show

Generalized Momentum $\quad \vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = \mu\dot{\vec{r}}$

Generalized Force $\quad \vec{F} = \frac{\partial L}{\partial \vec{r}} = -k\vec{r}$

$$\vec{F} = \dot{\vec{p}} \quad \Rightarrow \quad \mu\ddot{\vec{r}} = -k\vec{r}$$

Spring in Free Space

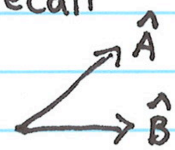
So $\ddot{\vec{r}} + \frac{\mu}{K}\vec{r} = \ddot{\vec{r}} + \omega^2\vec{r} = 0$

Solution:

$$\vec{r} = \vec{A}\cos(\omega t) + \vec{B}\sin(\omega t)$$

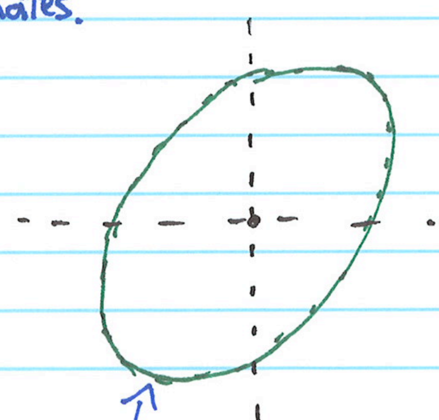
6 initial conditions.

This is written in 3 dimensions. But recall that $\vec{A}$ and $\vec{B}$ are constant vectors.
and.. Two vectors define a plane.



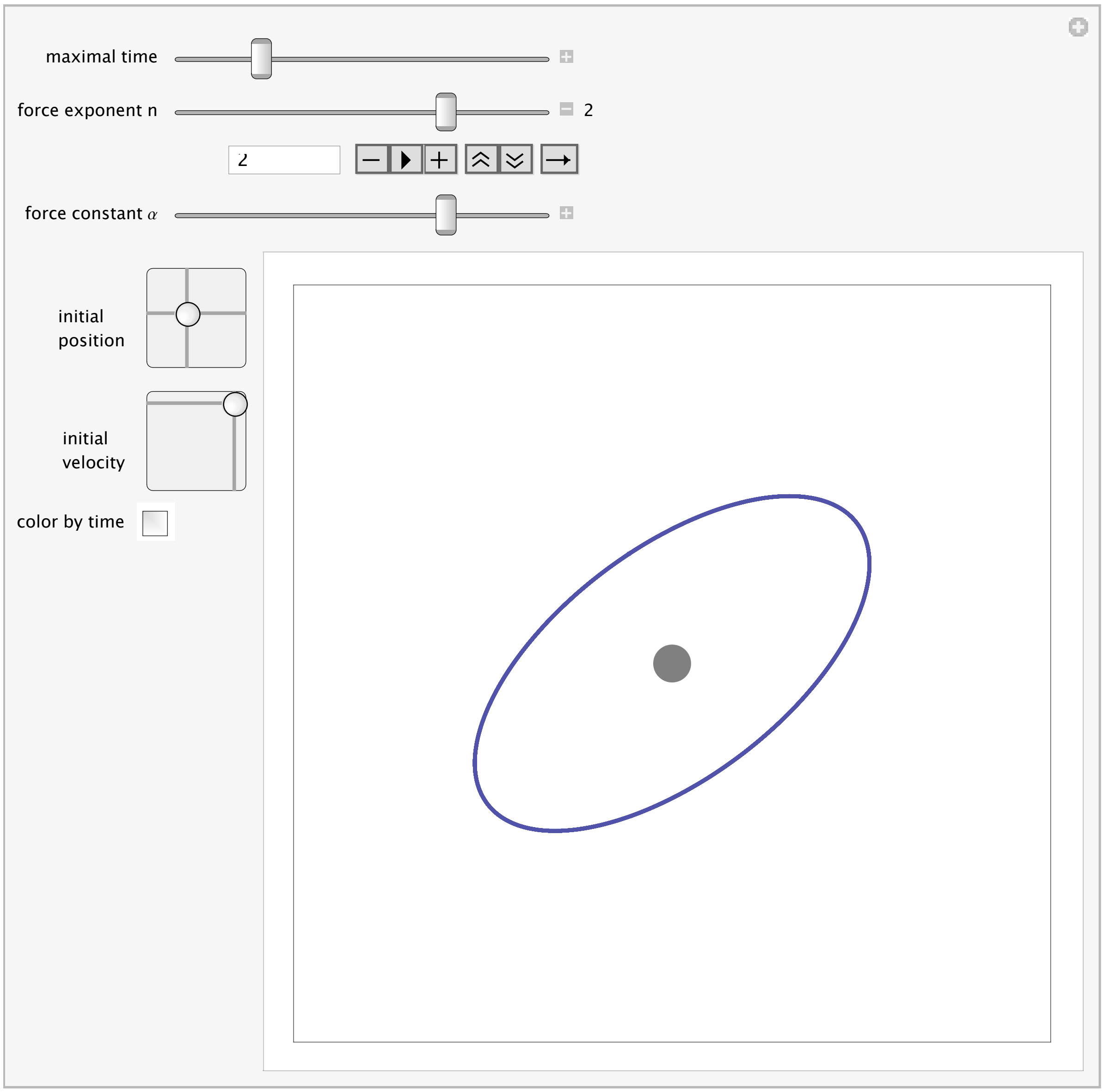
So the motion is in the plane defined by $\vec{A}$ and $\vec{B}$ and can be characterized using only 2 coordinates.

Why?



Ellipse with Origin at centre

Motion in a Central Field



This demonstrates a planar trajectory in a central field represented by a potential of the form $V(r) = \alpha\, r^n$. Motion in a central field is one of the classical problems of classical mechanics. Many interesting phenomena are exhibited by the motion of a small mass in a central field, like Bertrand's theorem, the fall into the center for strongly singular potentials, ….

Springs in Free Space

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Z}}\right)$$

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Spring in Free Space

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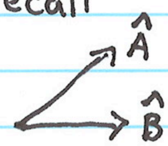

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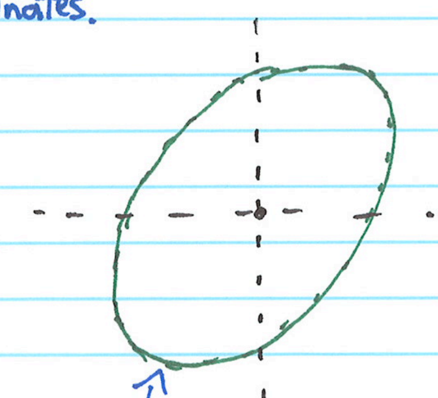
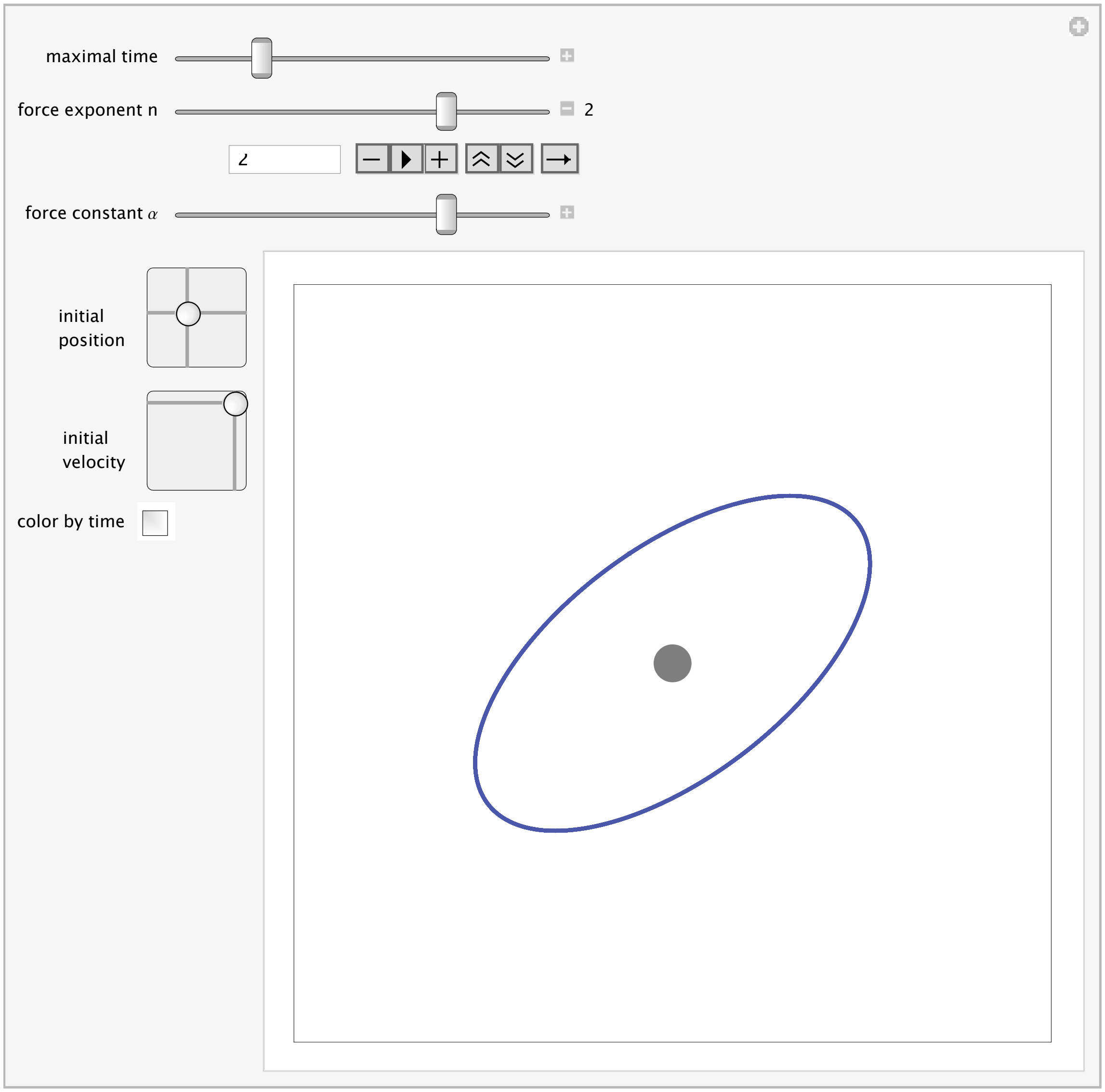
Why?



Ellipse with Origin at centre

Motion in a Central Field



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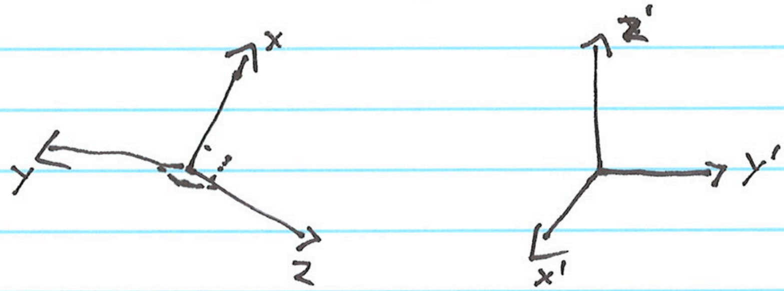
Spring in Free Space

There are no external forces or torques on the system

$\Rightarrow$ the system is rotationally invariant.



$\Rightarrow$ Total Angular Momentum $\vec{L}$ is conserved

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$
$$\vec{L} = m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2$$

Can show if $\vec{R} = 0$ C.O.M frame

$$\vec{L} = \vec{r} \times \mu \dot{\vec{r}} = \overrightarrow{\text{const}}$$

Direction $\hat{L}$ is constant. Defines plane.

Spring in Free Space

So generally $L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - \frac{1}{2} k \vec{r}^2$

We can choose to work in the Centre of Mass inertial frame $\vec{R} = 0 \quad \dot{\vec{R}} = 0$

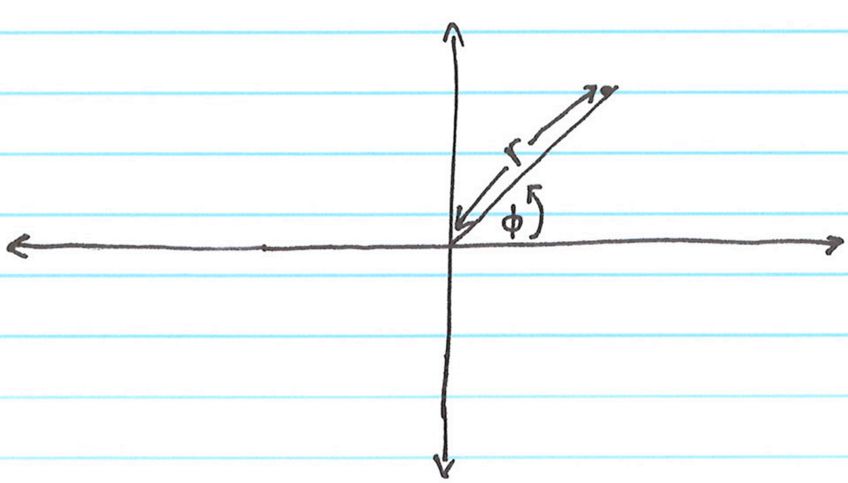
$$\Rightarrow \quad L_{CM} = \frac{1}{2} M \dot{\vec{R}}^2 = 0$$

$$L = \frac{1}{2} \mu \dot{\vec{r}}^2 - \frac{1}{2} k \vec{r}^2$$

We can further choose plane polar coordinates

$$\vec{r} = r \hat{r} \qquad \dot{\vec{r}}^2 = \dot{r}^2 + r^2 \dot{\phi}^2$$

$$\Rightarrow \quad L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\phi}^2) - \frac{1}{2} k r^2$$



Spring in Free Space

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}kr^2$$

ϕ equation:

$$\frac{\partial L}{\partial \phi} = 0 \quad \text{so} \quad \phi \text{ is ignorable}$$

and is associated with the conservation of

$$\tilde{p}_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = \ell$$

ℓ is the angular momentum
(points in the $\hat{z}$ direction out of plane)

r equation:

$$\tilde{F}_r = \frac{\partial L}{\partial r} = \mu r \dot{\phi}^2 - kr$$

$$\tilde{p}_r = \frac{\partial L}{\partial \dot{r}} = \mu \dot{r}$$

$$\therefore \quad \mu \ddot{r} + kr - \mu r \dot{\phi}^2 = 0$$

Spring in Free Space

Now $\mu r^2 \dot{\phi} = \ell$

: $\mu \ddot{r} + kr - \underbrace{\mu r \dot{\phi}^2}_{\text{centrifugal force}} = 0$

$\Rightarrow \quad \dot{\phi} = \dfrac{\ell}{\mu r^2}$ angular velocity is function of r

Radial equation:

Looks like 1D problem

$$\mu \ddot{r} + kr - \frac{\ell^2}{\mu r^3} = 0$$

Can now solve for $r(t)$
and then $\dot{\phi}(t)$
and then $\phi(t)$

1D: $\mu \ddot{r} = -\dfrac{dU_{eff}}{dr}$

$$\mu \ddot{r} + \frac{\partial U_{eff}}{\partial r} = 0$$

Spring in Free Space

Equivalent 1D problem

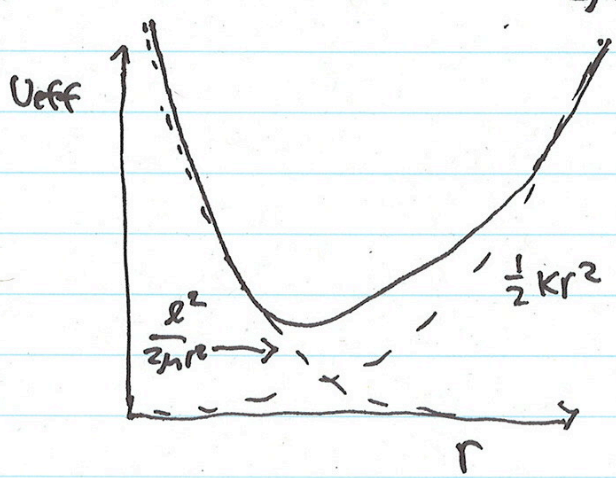
$$\mu\ddot{r} + kr - \mu r\left(\frac{\ell}{\mu r^2}\right)^2 = 0$$

$$\mu\ddot{r} + kr - \frac{\ell^2}{\mu r^3} = 0$$

$$\mu\ddot{r} + \frac{\partial}{\partial r}\underbrace{\left[\frac{1}{2}kr^2 + \frac{\ell^2}{2\mu r^2}\right]}_{= U(r)} = 0$$

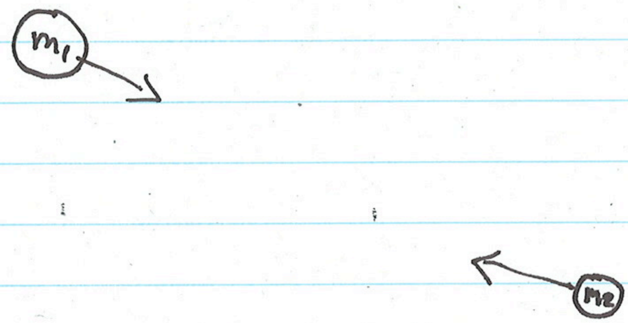
Same equation of motion with

$$U_{eff} = \frac{1}{2}kr^2 + \frac{\ell^2}{2\mu r^2}$$



Lagrangian Mechanics

Two Body Central Potential



Potential

$$U = U(|\vec{r}_1 - \vec{r}_2|) \overset{\text{we did}}{=} \frac{1}{2}k|\vec{r}_1 - \vec{r}_2|^2$$

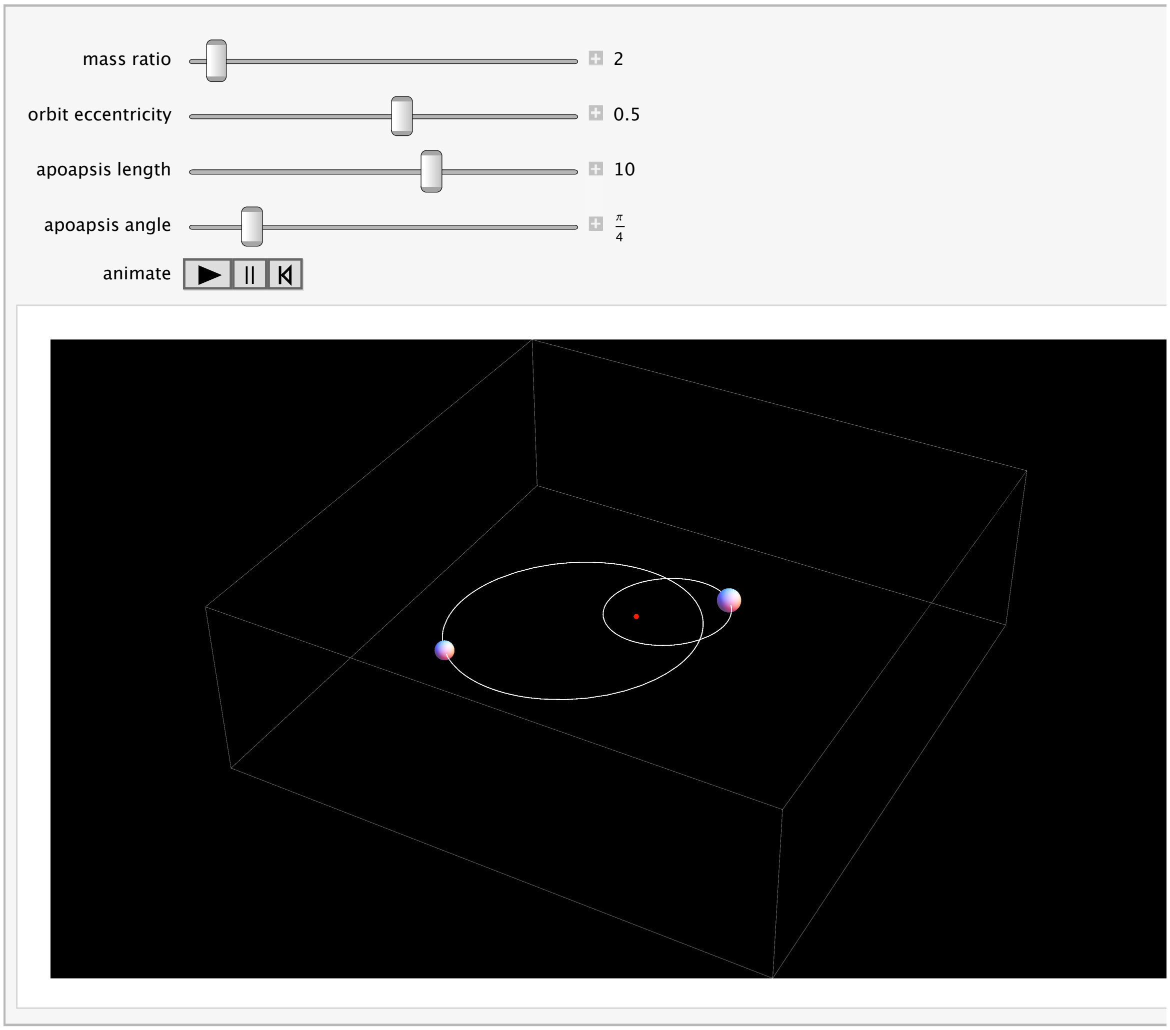
Everything carries through in the same way.

$$\frac{1}{2}kr^2 \rightarrow U(r)$$

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

Wolfram Demonstrations Project

The Celestial Two-Body Problem



Two celestial bodies interacting gravitationally can establish a stable system in which both trace out elliptical (or possibly circular) orbits about their mutual barycenter (center of mass), marked with a red dot. Each orbit individually follows Kepler's three laws of planetary motion.

Kepler's first law specifies that each orbit is an ellipse with one focus at the barycenter.

According to Kepler's second law, each orbit sweeps out equal areas in equal times. Thus for an eccentric orbit, the speed increases closer to the barycenter.

Lagrangian Mechanics

Two Body Central Potential $U(|\vec{r}_1 - \vec{r}_2|)$

Our solution in a harmonic potenial $\frac{1}{2}kr^2$ carries over provided we take $\frac{1}{2}kr^2 \rightarrow U(r)$

1. Solve using $\vec{r} = \vec{r}_1 - \vec{r}_2$ $\vec{R} = \frac{m_1}{m_1+m_2}\vec{r}_1 + \frac{m_2}{m_1+m_2}\vec{r}_2$ and find

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \qquad \mu = \frac{m_1 m_2}{M} \quad M = m_1 + m_2$$

2. In the C.O.M frame $\vec{R} = 0$ $\dot{\vec{R}} = 0$

$$L = L_{cm} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$

3. Motion constrained to a plane perpendicular to the angular momentum vector in the C.OM.

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

Use plane-polar coordinates

Lagrangian Mechanics

Two Body Central Potential

ϕ: $\quad \dfrac{\partial L}{\partial \phi} = 0$

$$\Rightarrow \quad \ell = \mu r^2 \dot{\phi} = \text{const.}$$

r: $\quad \dfrac{\partial L}{\partial r} = \mu r \dot{\phi}^2 - \dfrac{\partial U}{\partial r}$

$$\frac{\partial L}{\partial \dot{r}} = \mu \dot{r}$$

$$\Rightarrow \quad \mu \ddot{r} = \mu r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Equivalent 1D problem

defined given some value ℓ

$$\text{For} \quad U_{eff}(r) = U(r) + \frac{\ell^2}{2\mu r^2}$$

$$\mu \ddot{r} = -\frac{\partial U_{eff}}{\partial r}$$

Energy

$$E = T + U$$

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + U(r)$$

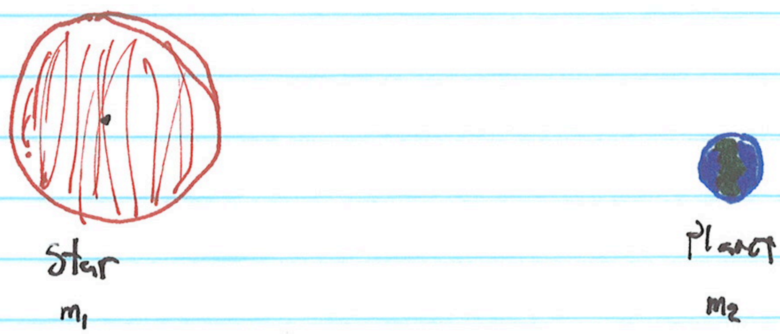
$$E = \frac{1}{2}\mu \dot{r}^2 + \underbrace{\frac{\ell^2}{2\mu r^2} + U(r)}$$

Equivalent 1D problem with same E $\longrightarrow$ $E = \frac{1}{2}\mu \dot{r}^2 + U_{eff.}(r)$

Lagrangian Mechanics

Kepler Problem

Not to Scale!



a) What is $U(r)$? $U(r) = -\frac{Gm_1m_2}{r} = -\frac{\gamma}{r}$

b) Lagrange equations

c) $U_{eff}(r)$ $\rightarrow$ sketch Equivalent 1D problem

d) What kind of motion allowed for fixed E?

$$\mu\ddot{r} = -\frac{\partial U_{eff}}{\partial r}$$

Lagrangian Mechanics

Kepler Problem

$$U(r) = -\frac{Gm_1 m_2}{r} = -\frac{\gamma}{r}$$

So
$$L = T - U$$
$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{\gamma}{r}$$

ϕ: $\quad \ell = \mu r^2 \dot{\phi}$

r: $\quad \mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{\gamma}{r^2}$

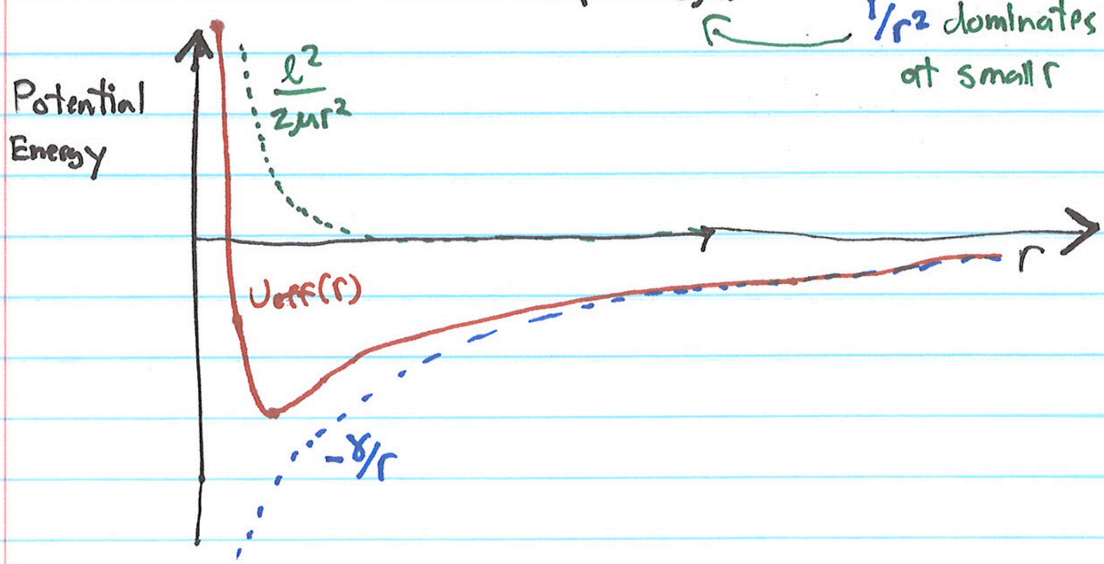
$$\frac{\partial L}{\partial r} = \mu r\dot{\phi}^2 - \frac{\gamma}{r^2} = \mu r\left(\frac{\ell}{\mu r^2}\right)^2 - \frac{\gamma}{r^2}$$

Effective Potential

$$U_{eff}(r) = -\frac{\gamma}{r} + \frac{\ell^2}{2\mu r^2}$$

$1/r$ dominates at larger r

$1/r^2$ dominates at small r



Kepler Problem

For a potential $U(r) = -\gamma/r$

The effective potential at fixed angular momentum ℓ is:

$$U_{eff}(r) = \frac{\ell^2}{2\mu r^2} + U(r)$$

$$U_{eff}(r) = \frac{\ell^2}{2\mu r^2} - \frac{\gamma}{r}$$

Effective Potential

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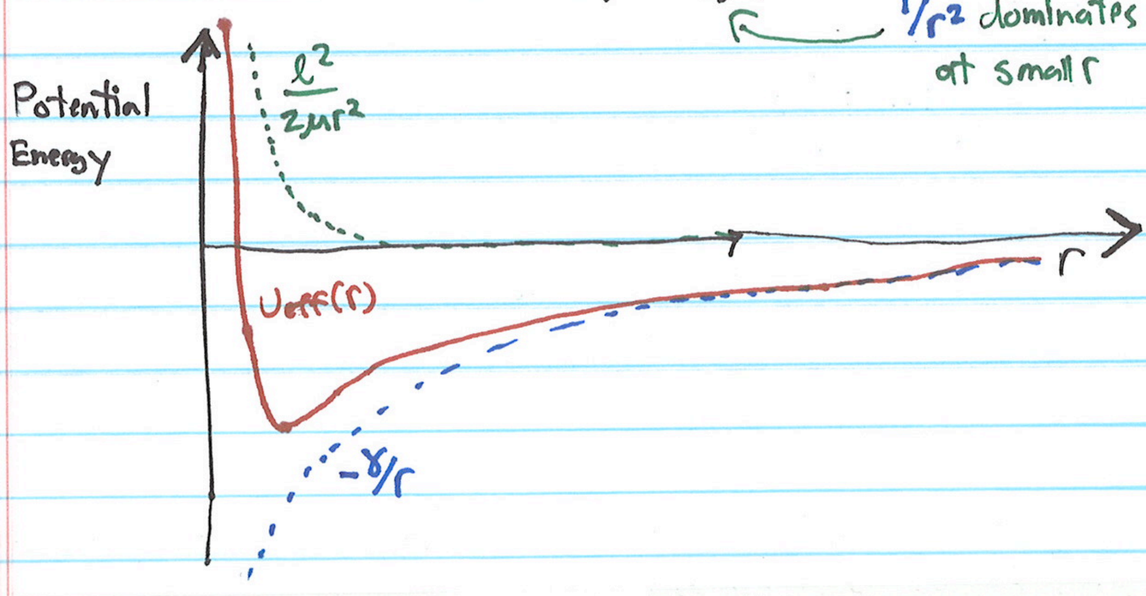
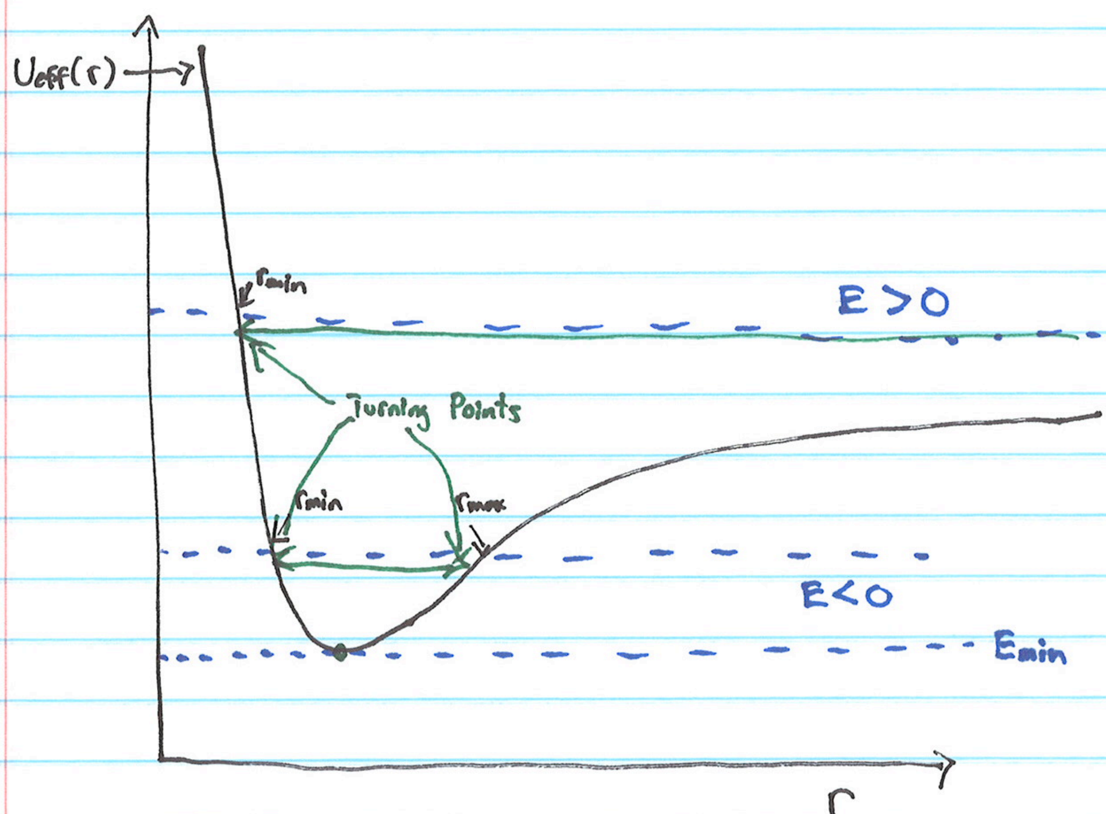


Lagrangian Mechanics

Kepler Problem



Unbound $\infty > r > r_{min}$

Bound $r_{max} \geq r \geq r_{min}$

Solve $E = U_{eff}(r_{min})$ or $E = U_{eff}(r_{max})$ to find r_{min}, r_{max}