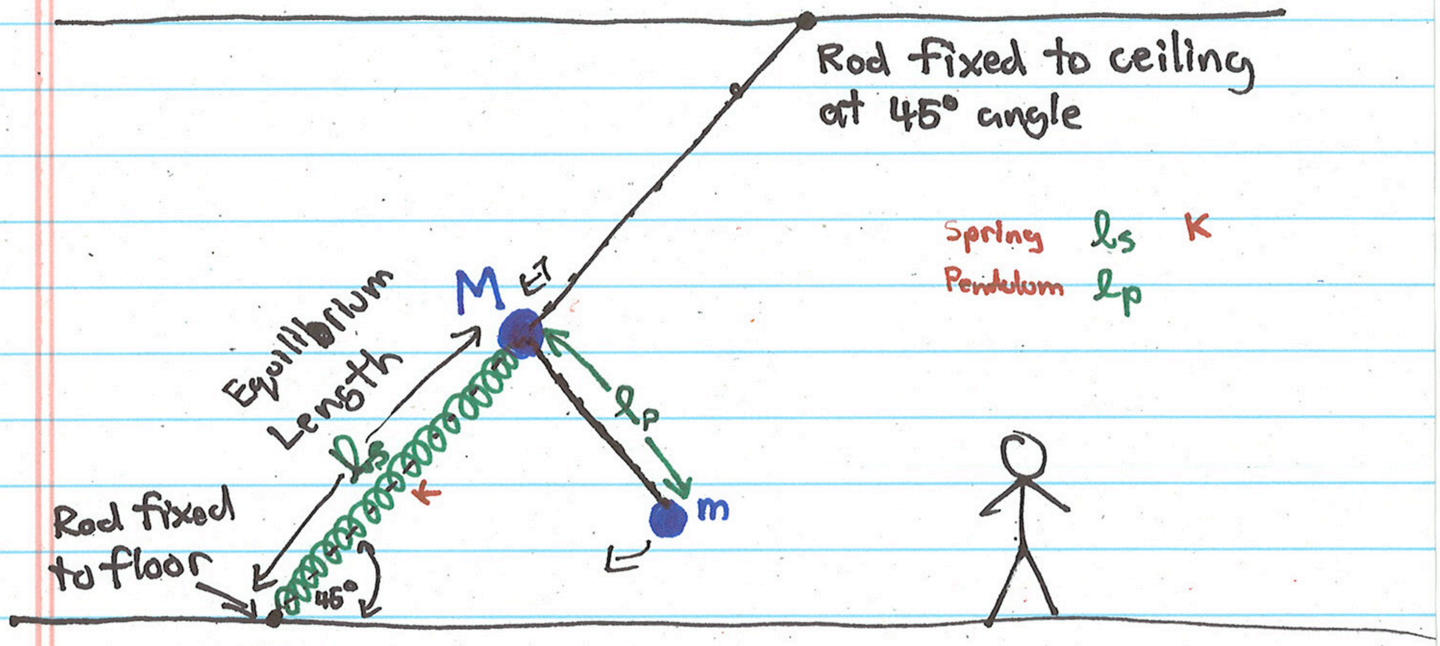
Lagrangian Mechanics cont.

Example: The Point Grey Pendulum



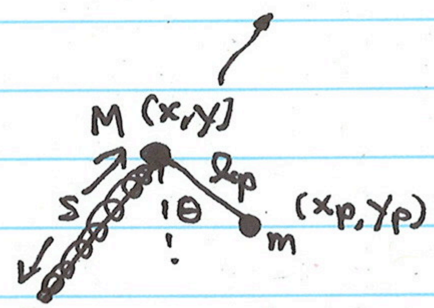
a) Choose generalized coordinates and write down the Lagrangian

b) Write down the equations of motion.

c) Easier or ?

Lagrangian Mechanics cont.

Example: The Point Grey Pendulum



a) Choose length along rod "s" and the pendulum angle "θ"

Kinetic $$T = \underbrace{\tfrac{1}{2}M(\dot{x}^2+\dot{y}^2)}_{T_s} + \underbrace{\tfrac{1}{2}m(\dot{x}_p^2+\dot{y}_p^2)}_{T_p}$$

Now $s = \sqrt{x^2+y^2} = \sqrt{2}\,x$ since $y = x$

Spring so $T_s = \tfrac{1}{2}M(\dot{x}^2+\dot{y}^2) = \tfrac{1}{2}M\dot{s}^2$

Pendulum
$$x_p = x + l_p\sin\theta \qquad y_p = x - l_p\cos\theta$$
$$\dot{x}_p = \dot{x} + l_p\cos\theta\,\dot{\theta} \qquad \dot{y}_p = \dot{x} + l_p\sin\theta\,\dot{\theta}$$

$$\begin{aligned}\dot{x}_p^2+\dot{y}_p^2 &= 2\dot{x}^2 + 2l_p[\cos\theta+\sin\theta]\dot{x}\dot{\theta} + l_p^2\dot{\theta}^2\\ &= 2\dot{x}^2 + 2l_p\cos(\theta-\tfrac{\pi}{4})\sqrt{2}\,\dot{x}\dot{\theta} + l_p^2\dot{\theta}^2\\ &= \dot{s}^2 + 2l_p\cos(\theta-\tfrac{\pi}{4})\,\dot{s}\dot{\theta} + l_p^2\dot{\theta}^2\end{aligned}$$

Lagrangian Mechanics cont.

Example: Point Grey Pendulum cont.

so $$T = \frac{1}{2}(M+m)\dot{s}^2 + \frac{1}{2} m \ell_p^2 \dot{\theta}^2 + m \ell_p \cos\left(\theta - \frac{\pi}{4}\right)\dot{s}\dot{\theta}$$

Potential

$$U = Mgy + mgy_p + \frac{1}{2}K(s - \ell_s)^2$$

$$\Rightarrow U = Mgx + mg(x - \ell_p \cos\theta) + \frac{1}{2}K(s - \ell_s)^2$$

$$x = \frac{s}{\sqrt{2}}$$

$$U = \frac{1}{\sqrt{2}}(M+m)gs + \frac{1}{2}K(s-\ell_s)^2 - mg\ell_p\cos\theta$$

$$\therefore L = \frac{1}{2}(M+m)\dot{s}^2 + \frac{1}{2}m\ell_p^2\dot{\theta}^2 + m\ell_p\cos\left(\theta - \frac{\pi}{4}\right)\dot{s}\dot{\theta} - \frac{1}{\sqrt{2}}(M+m)gs - \frac{1}{2}K(s-\ell_s)^2 + mg\ell_p\cos\theta$$

Lagrangian Mechanics cont.

Example: Point Grey Pendulum cont.

s Equation:

$$\tilde{p}_s = \frac{\partial L}{\partial \dot{s}} = (M+m)\dot{s} + m\ell_p \cos(\theta - \tfrac{\pi}{4})\dot{\theta}$$

$$\tilde{F}_s = \frac{\partial L}{\partial s} = -\frac{1}{\sqrt{2}}(M+m)g - K(s-\ell_s)$$

$$\frac{\partial L}{\partial s} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right)$$

$$\Rightarrow -\frac{1}{\sqrt{2}}(M+m)g - K(s-\ell_s) = (M+m)\ddot{s} + m\ell_p\cos(\theta-\tfrac{\pi}{4})\ddot{\theta} - m\ell_p \sin(\theta - \tfrac{\pi}{4})\dot{\theta}^2$$

$$(M+m)\ddot{s} + m\ell_p\cos(\theta-\tfrac{\pi}{4})\ddot{\theta} - m\ell_p\sin(\theta-\tfrac{\pi}{4})\dot{\theta}^2 + K(s-\ell_s) + \frac{1}{\sqrt{2}}(M+m)g = 0$$

θ Equation:

$$\tilde{p}_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell_p^2\dot{\theta} + m\ell_p\cos(\theta-\tfrac{\pi}{4})\dot{s}$$

$$\tilde{F}_\theta = \frac{\partial L}{\partial \theta} = -m\ell_p\sin(\theta-\tfrac{\pi}{4})\dot{s}\dot{\theta} - mg\ell_p\sin\theta$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$$

Lagrangian Mechanics cont.

Example: The Point Grey Pendulum cont.

$$-m l_p \sin(\theta - \tfrac{\pi}{4})\dot{s}\dot{\theta} - mg l_p \sin\theta$$
$$= m l_p^2 \ddot{\theta} + m l_p \cos(\theta - \tfrac{\pi}{4})\ddot{s}$$
$$- m l_p \sin(\theta - \tfrac{\pi}{4})\dot{s}\dot{\theta}$$

$$\therefore \quad l_p \ddot{\theta} + \cos(\theta - \tfrac{\pi}{4})\ddot{s} + g\sin\theta = 0$$

let $\alpha = \frac{m}{m+M}$ $\quad \omega^2 = \frac{K}{m+M}$

$$\Rightarrow \quad \ddot{s} + \alpha l_p \cos(\theta - \tfrac{\pi}{4})\ddot{\theta} - \alpha l_p \sin(\theta - \tfrac{\pi}{4})\dot{\theta}^2$$
$$+ \omega^2 (s - l_s) + \tfrac{1}{\sqrt{2}} g = 0$$

$$l_p \ddot{\theta} + \cos(\theta - \tfrac{\pi}{4})\ddot{s} + g\sin\theta = 0$$

Can insted use $\quad \phi = \theta - \frac{\pi}{4} \quad\quad u = s - l_s$

$$\ddot{u} + \alpha l_p \cos\phi\, \ddot{\phi} - \alpha l_p \sin\phi\, \dot{\phi}^2 + \omega^2 u + \tfrac{1}{\sqrt{2}} g = 0$$

$$l_p \ddot{\theta} + \cos\phi\, \ddot{u} + g\sin(\phi + \tfrac{\pi}{4}) = 0$$

Lagrangian Mechanics

Ignorable Coordinates

Given some choice of generalized coordinates q_i $(i=1,\ldots,n)$ we in general have

$$L = L(q_1, \dot{q}_1, q_2, \dot{q}_2, \ldots, q_n, \dot{q}_n)$$

Generalize Force $\quad \tilde{F}_i = \frac{\partial L}{\partial q_i}$

Generalized Momentum $\quad \tilde{p}_i = \frac{\partial L}{\partial \dot{q}_i}$

E-L (Lagrange Equation) $\Rightarrow \quad \tilde{F}_i = \frac{d}{dt}(\tilde{p}_i)$

If a Lagrangian is independent of a particular coordinate q_j, then the coordinate is called an ignorable coordinate and the corresponding generalized momenton $\tilde{p}_j$ is constant.

Lagrangian Mechanics cont.

Example: Projectile Motion

In general projectile motion requires 3 coordinates $q_1 = x \quad q_2 = y \quad q_3 = z$

$$U = mgz$$
$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

Dynamical Coordinate $\Rightarrow$ z: $\frac{\partial L}{\partial z} = -mg = \frac{d}{dt}(m\dot{z}) = m\ddot{z}$

Ignorable Coordinates $\Rightarrow$ x, y

$$F_x = \frac{\partial L}{\partial x} = 0 \Rightarrow p_x = m\dot{x} = \frac{\partial L}{\partial \dot{x}} \text{ constant}$$
$$F_y = \frac{\partial L}{\partial y} = 0 \Rightarrow p_y = m\dot{y} = \frac{\partial L}{\partial \dot{y}} \text{ constant}$$

* Note: System (and Lagrangian) are invariant under translations in the x and y coordinates

Example of Noether's theorem.

Lagrangian Mechanics

Noether's Theorem

(Emmy Noether 1882-1935)

Every continuous symmetry of the Lagrangian is associated with a $\boxed{\text{conserved "charge"}}$

Consider $L = L(q_i, \dot{q}_i, t)$ $\quad q_i = \{q_1, q_2, \ldots, q_n\}$

and imagine a one-parameter family of transformations

$$q_i \rightarrow \bar{q}_i(q_i, \xi)$$

ie $\quad (x, y, z) \rightarrow (x+\xi, y, z) = (\bar{x}, \bar{y}, \bar{z})$

If the Lagrangian is invariant under this transformation then

$$L(q_i, \dot{q}_i, t) = L(\bar{q}_i, \dot{\bar{q}}_i, t)$$

ie. $\quad L(x, \dot{x}, t) = L(\bar{x}, \dot{\bar{x}}, t)$