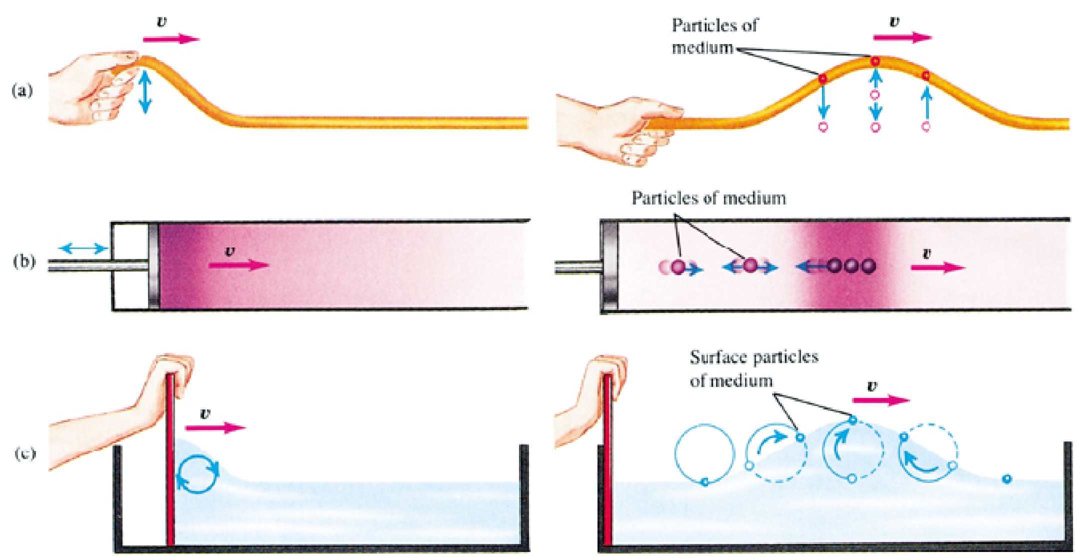
Chapter 19 : Mechanical Waves

There are many different kinds of waves in nature, sound, light and even particles exhibit wave-like properties. Mechanical waves require a medium in which they propagate. Without the wave the medium is in equilibrium (e.g. a string under tension). In the medium is disturbed away from equilibrium this costs energy. Such a disturbance will propagate in the medium in the form of waves.
Some properties of mechanical waves:

1. waves transport energy from one part of the medium to another.
2. the speed of propagation or wave speed v depends on the medium.
3. in a wave there is local motion in the medium about equilibrium but there is no transport of matter. (note this is quite unlike energy transport when a ball is thrown in the air.)
4. If the local motion of the medium is transverse to motion of the wave as the wave moves through then the wave is called a transverse wave . (wave on a string) .
Similarly if the local motion of the medium is longitudinal to the wave motion the wave is a called a longitudinal wave.



Periodic Wave

In a wave pulse the medium is displaced away from equilibrium at any particular position only for a short time when the pulse is passing.

One can also create a continuous wave disturbance called a periodic wave in which the displacement of the atoms varies sinusoidally both as a function of space and time.

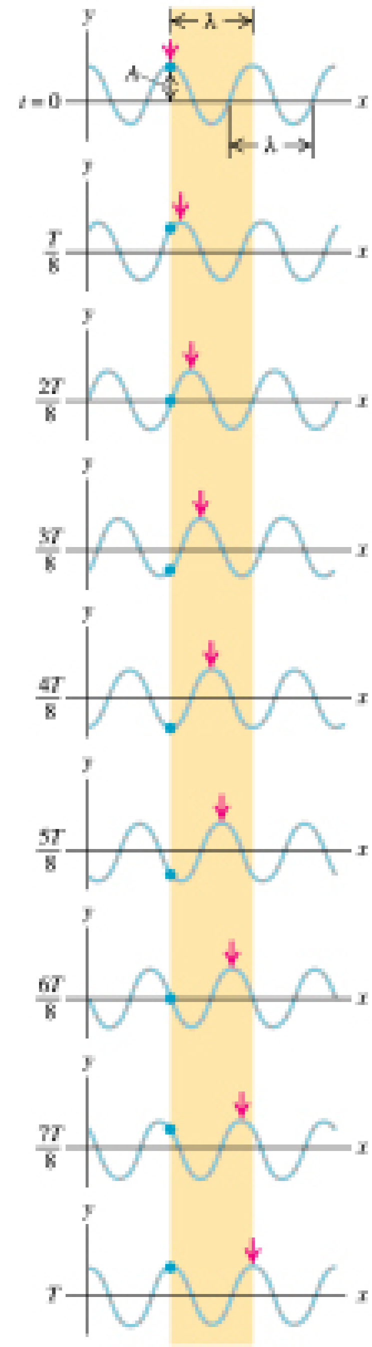
$$y(x,t) = A\sin(\omega t - \frac{2\pi x}{\lambda}); \quad \text{Eqn*}$$

In other words at any position the medium moves in simple harmonic motion with a angular frequency ω and period $T=1/f=2\pi/\omega$. (see blue dot)

Also, at any particular time the displacement varies sinusoidally with a wavelength λ. Note in one period T the crest moves one wavelength. Thus the wave propagates with a speed $v=\lambda/T=f\lambda$. Rewriting Eqn *

$$y(x,t) = A\sin(\omega t - \frac{f 2\pi x}{f\lambda}) = A\sin(\omega t - \frac{\omega x}{v}) = A\sin\omega(t - \frac{x}{v})$$

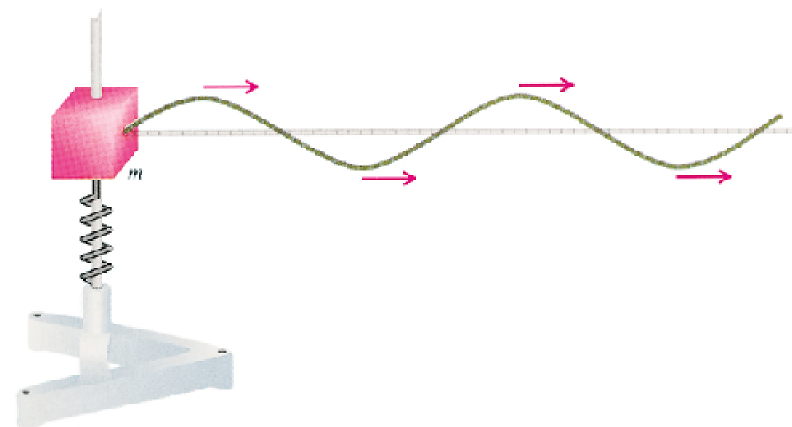
$$= A\sin(\omega t - kx); \text{ where } k = \frac{2\pi}{\lambda} \text{ is called the wave number.}$$



This “wave function” describes a sinusoidal wave moving in the +ve x- direction. What would the wave function be for a wave moving is –x direction?
If one follows a particular crest of the wave then the argument of the sin function , $\omega t - kx_{crest} = \pi/2$

$$\frac{dx_{crest}}{dt} = \frac{\omega}{k} = f\lambda; \text{as before}$$

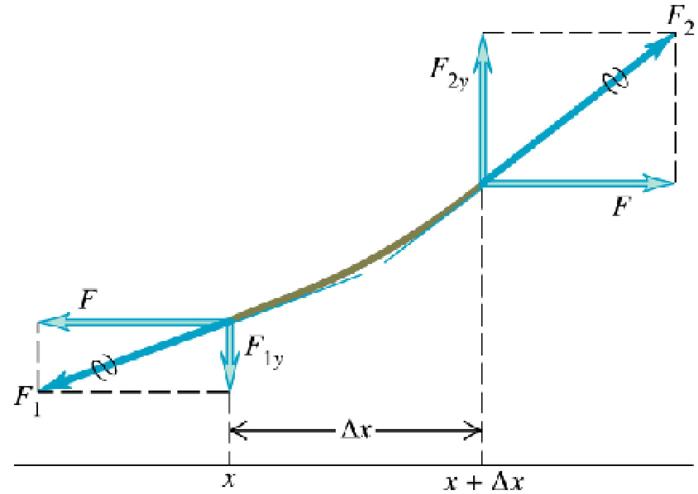
Example. The end of a string is moved up and down sinusoidally with a frequency f=2Hz with an amplitude 0.075 m such that the displacement is zero at t=0.
The resulting periodic wave has a velocity of 12.0m/s. (a) What are the amplitude wavenumber, angular frequency and wavelength? (b)
What is the displacement as a function of time 3 m down from the end?



Wave function and speed of a wave

Consider a small element of the wave of length Δx on a string with mass per unit length μ which is under tension F. In equilibrium without any disturbance the string lies on the x axis such y(x,t)=0. If there is disturbance we can show that wave equation is satisfied for each and every element on the string.

$$\frac{F_{1y}}{F} = -\left(\frac{\partial y}{\partial x}\right)_x \quad ; \quad \frac{F_{2y}}{F} = +\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x}$$



Applying F=ma on this element leads to:

$$F_{total} = F_{1y} + F_{2y} = m \quad a$$

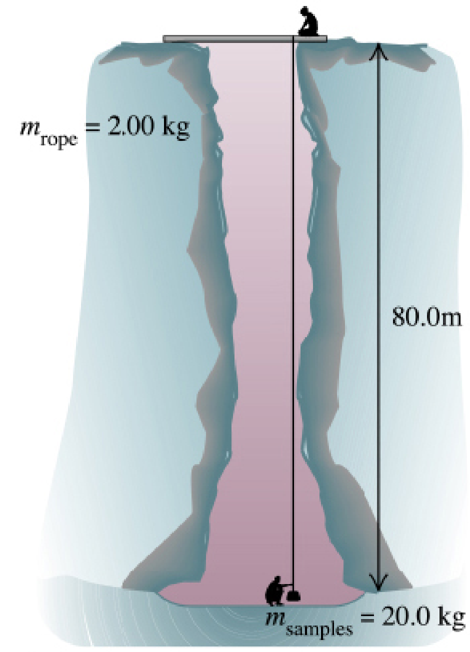
$$F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = \mu \Delta x \frac{\partial^2 y}{\partial t^2}$$

$$F\frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2}; \text{ or } \quad \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}; \text{ Eqn*; Wave equation}$$

Any disturbance of the string (with no external forces) can be described by a wave function y(x,t) which must satisfy the above wave equation. In particular Asin(ωt–kx) will only be a valid solution if the wave speed $v = f\lambda = \omega/k = (F/\mu)^{1/2}$. Note the wave speed is only a function of the string tension and mass per unit length. It doesn't depend on ω (or f), k(or λ) or A. Waves with different frequency must propagate with the same speed. If the frequency of a wave is doubled what happens to the wavelength?

Example

How long does it take for a wave pulse to travel up the rope?

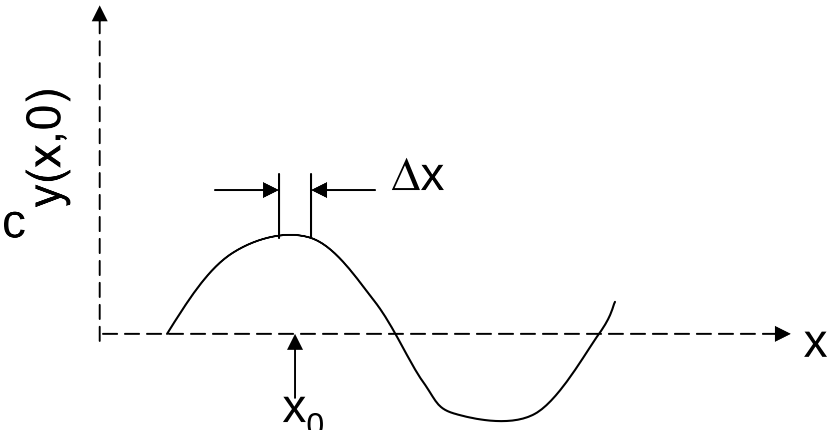


Wave power and energy density for periodic wave on a string

Consider a small element Δx at a position x_0 on a string with periodic wave

$$y(x,t) = A\sin(\omega t - kx)$$

$$y(x_0,t) = A\sin(\omega t - kx_0) = A\sin(\omega t + \phi)$$



The element is undergoing simple harmonic motion and therefore the time averaged energy of this small element :

$$\Delta E = K + U = \frac{1}{2} m v_{\max}^2 \text{; where } v_{\max} \text{ is the maximum velocity of the element}$$

$$= \frac{1}{2}\mu \Delta x (A\omega)^2$$

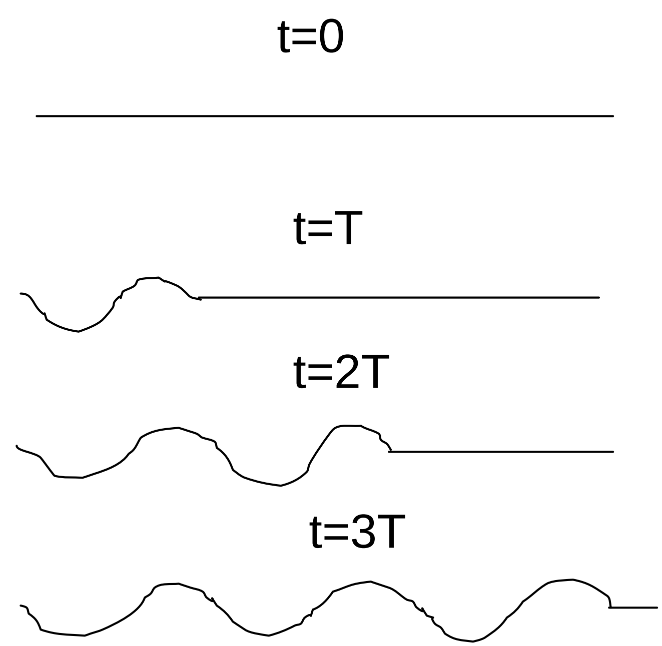
Thus the energy density (kinetic plus potential) or energy per unit length $\frac{\Delta E}{\Delta x} = \frac{1}{2}\mu (A\omega)^2$

The energy in one wavelength is $\frac{\Delta E}{\Delta x}\lambda = \frac{1}{2}\mu\lambda (A\omega)^2$

The transmitted energy in one period T or the average wave power is then

$$P = \frac{\Delta E}{\Delta x}\frac{\lambda}{T} = \frac{1}{2}\mu(\lambda f)(A\omega)^2 = \frac{1}{2}\mu v (A\omega)^2$$

$$\text{where } v = \left[\frac{F}{\mu}\right]^{1/2}$$



Principle of Superposition

This principle states that if one has two separate waves (pulsed or continuous) propagating on a string (medium) described by wave functions $y_1(x,t)$ and $y_2(x,t)$ then the total displacement of the string is given by a combined wave function $y(x,t)$ which is just a linear sum of the two separate wave functions:

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

Show this function satisfies the wave equation.

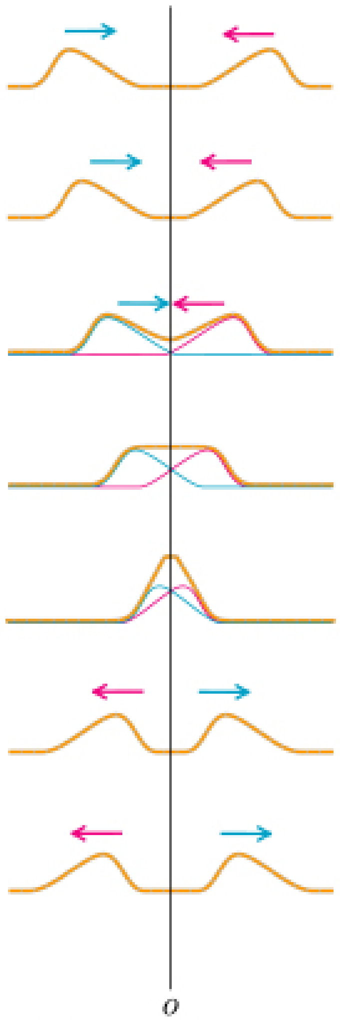
$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2};\left(\frac{d^2 y}{dx^2} = \frac{\mu}{F}\frac{d^2 y}{dt^2}\ \text{if you don't like partial derivatives}\right)$$

Why must this be true? Based of what you know about the simple harmonic oscillator under what conditions would you expect the wave equation and principle of superposition to breakdown for a string?

1. low frequency
2. high frequency
3. small amplitude
4. large amplitude

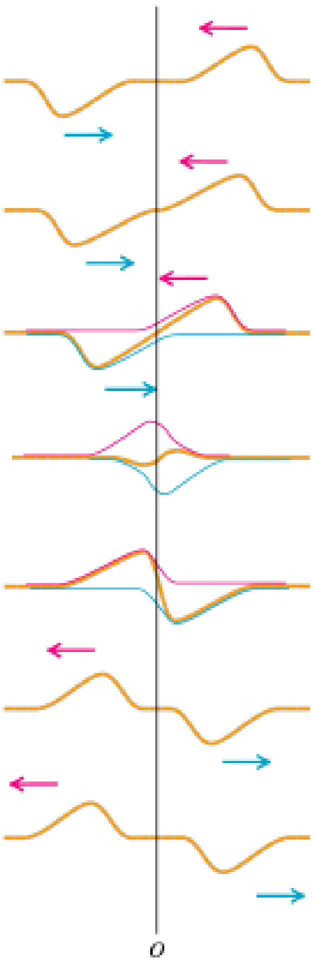
Principle of superposition applied to two opposite propagating pulses

case 1:
amplitudes same sign



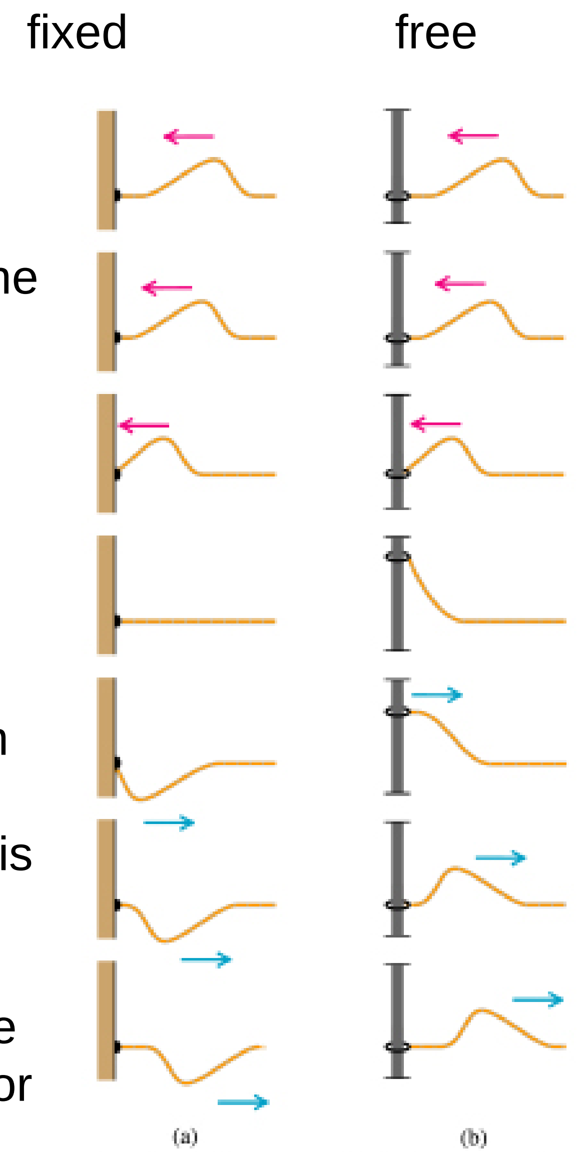
What is the wave speed
in these two cases?

case 2:
amplitudes of opposite sign



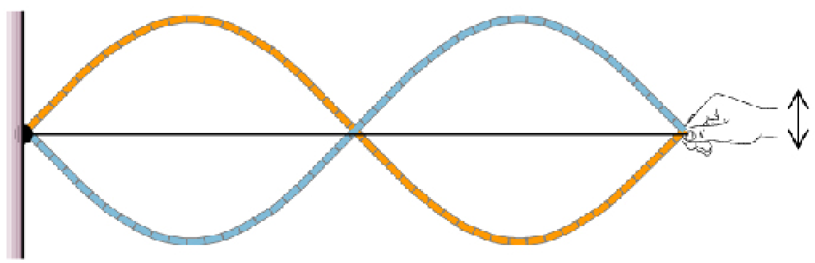
Reflection at a Boundary

For waves on a string there are two different kinds of reflections depending on the boundary conditions.
(a) For a fixed boundary the reflected wave has the opposite sign. It is as if a second wave was coming from the other side of the wall and arriving at the boundary at the same time as the real wave. Since it has the opposite sign the amplitude at the boundary is zero (by the principle of superposition)as it must be. Any point where the amplitude of the wave is always zero is called a node in the wave function. The wave must have a node at the wall position.
(b) At a free boundary the string is free to move up and down. Since the downward restoring force only comes from one side the amplitude is twice the amplitude of the incoming wave when the wave reaches the boundary. In this case the reflected wave has the same sign as the incident wave. At the boundary the amplitude of the string is at maximum. Any point where the amplitude of the wave is at a maximum is called antinode in the wave function. For a free boundary there is an antinode at the wall position.



Standing Waves

Consider a string which is fixed at both ends. One can create a wavefunction that doesn't appear to be moving in either direction. Every part of the spring undergoes SHM as in a propagating periodic wave. How is the motion different than in a propagating periodic wave? What do you notice about the amplitude as function of position?



One can also construct a standing wave pattern mathematically as a superposition of two periodic waves traveling in opposite directions with amplitudes which are equal in magnitude but opposite in sign.

$$y(x,t) = A[\sin(\omega t + kx) - \sin(\omega t - kx)]$$

using $\sin(\mathrm{a}+\mathrm{b}) = \sin(\mathrm{a})\cos(\mathrm{b}) + \cos(\mathrm{a})\sin(\mathrm{b})$

$$y(x,t) = A[\sin\omega t\cos kx + \cos\omega t\sin kx - \sin\omega t\cos kx + \cos\omega t\sin kx]$$

$$= 2A\sin kx\cos\omega t;$$

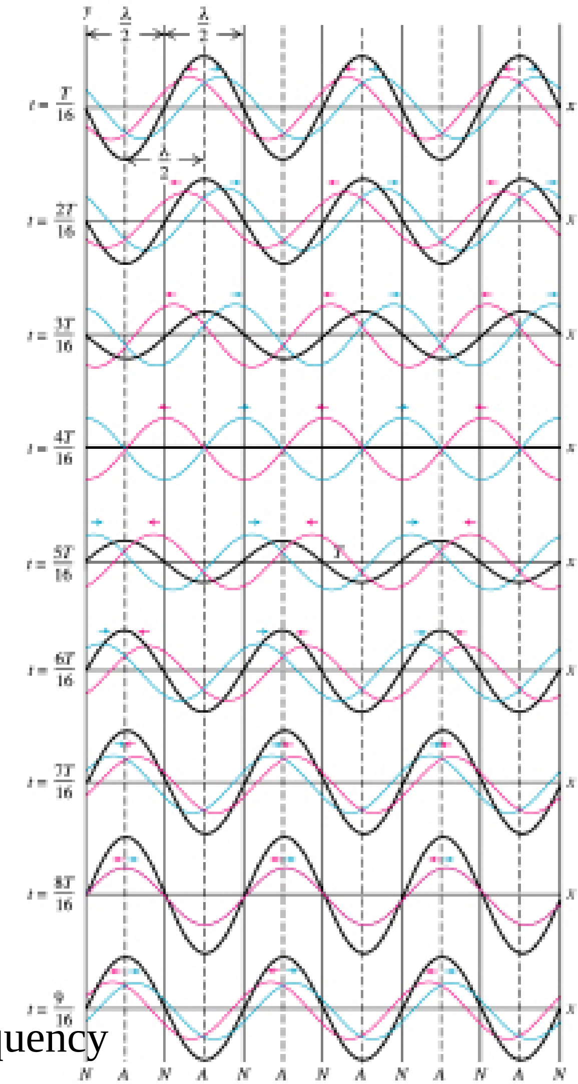
Where are the nodes in this function?

Now consider a string of length L, under tension F with mass per unit length μ which is fixed of both ends. Any standing wave.

$$y(x,t) = 2A\sin kx\cos\omega t;$$

will satisfy the wave equation

provided $\omega = vk$ where $v = \sqrt{F/\mu}$



However, only standing waves with particular values of k also satisfy the boundary conditions that there be a node at each end. For this to be true : $\dfrac{n\lambda}{2} = L;$ or $k = \dfrac{2\pi}{\lambda} = \dfrac{n\pi}{L}$

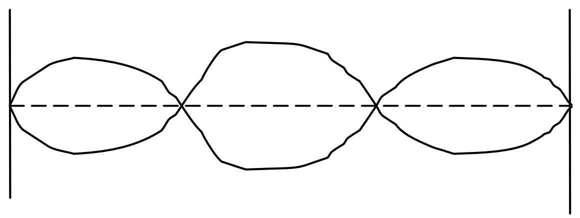
$$\text{or } y(x,t) = 2A\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{vn\pi}{L}t\right); \quad n = 1,2,3,4...$$

The integer n can be used to label the so called normal modes of vibration.

$$f_n = \frac{\omega_n}{2\pi} = \frac{vn}{2L} = nf_1; \text{ where } f_1 = \frac{v}{2\text{L}} \text{ is called the fundamental frequency}$$

Thus the string can only vibrate at discrete frequencies determined by the condition that a half integral number of wavelengths must equal the string length L.
What happens if the string has two parts?

Example: Consider a 50 cm string of mass 1.5g which is fixed at both ends. There is a standing wave on the string with a vibration frequency of 1200 Hz. Besides the ends there are two nodes in the string. What is the string tension?



Example

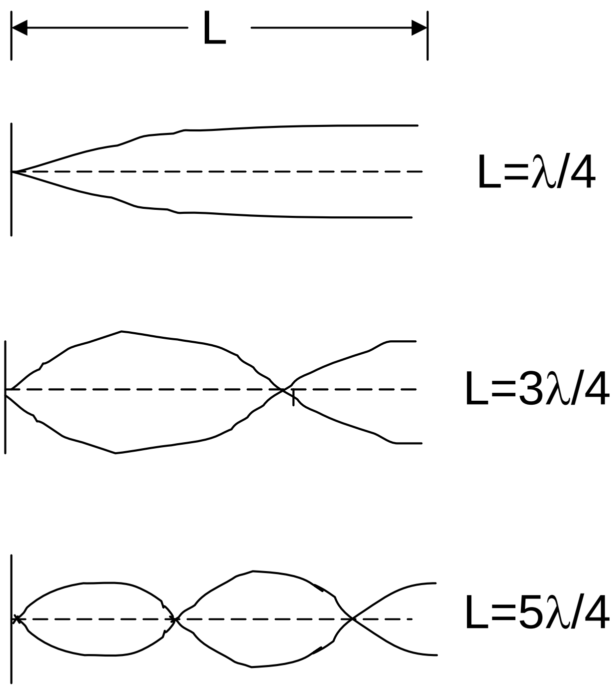
A 75 cm string is fixed at both ends. There are two normal modes of vibration with frequencies of f_a=420Hz and f_b=315 Hz and no frequencies in between. What is a wave speed?

Standing Waves with Open Boundary Conditions

Consider a string of length L, tension F, mass per unit length μ which is fixed at one end and free to move or open at the other.   What type of standing wave is possible with these boundary conditions? The form of the standing wave will be the same as before and since it must satisfy the wave equation.

$$y(x,t) = 2A\sin kx \cos \omega t; \text{ with the restriction } \omega = vk \text{ where } v = \sqrt{F/\mu}$$

However the boundary conditions are different. There must be a node at one end and an antinode at the other end.



$$L = (2n-1)\frac{\lambda}{4}; \text{ or } k = \frac{2\pi}{\lambda} = \frac{(2n-1)\pi}{2L}; \text{n} = 1,2,3,4$$

only standing waves where L is an odd multiple of $\lambda/4$ are possible

If the wave velocity is 10m/s and string is 2 m long what is the fundamental frequency?

What is the frequency of the second harmonic (second lowest frequency) if the string is open at both ends?

Example. Consider a metal rod of length L. Let y(x,t) be the transverse displacement of the rod away from equilibrium. y(x,t) satisfies the wave equation because each element of the rod experiences a restoring force proportional to both d^2y/dx^2 and the stiffness of the rod. This is the similar to a string under tension.

Suppose one holds the rod at its midpoint. Then you create a transverse standing wave. If the lowest frequency for transverse vibration is f_1 what is the velocity for transverse wave propagation? What is the lowest frequency of vibration if the rod is held at a distance L/4 from an end?

Normal Modes and Standing Waves

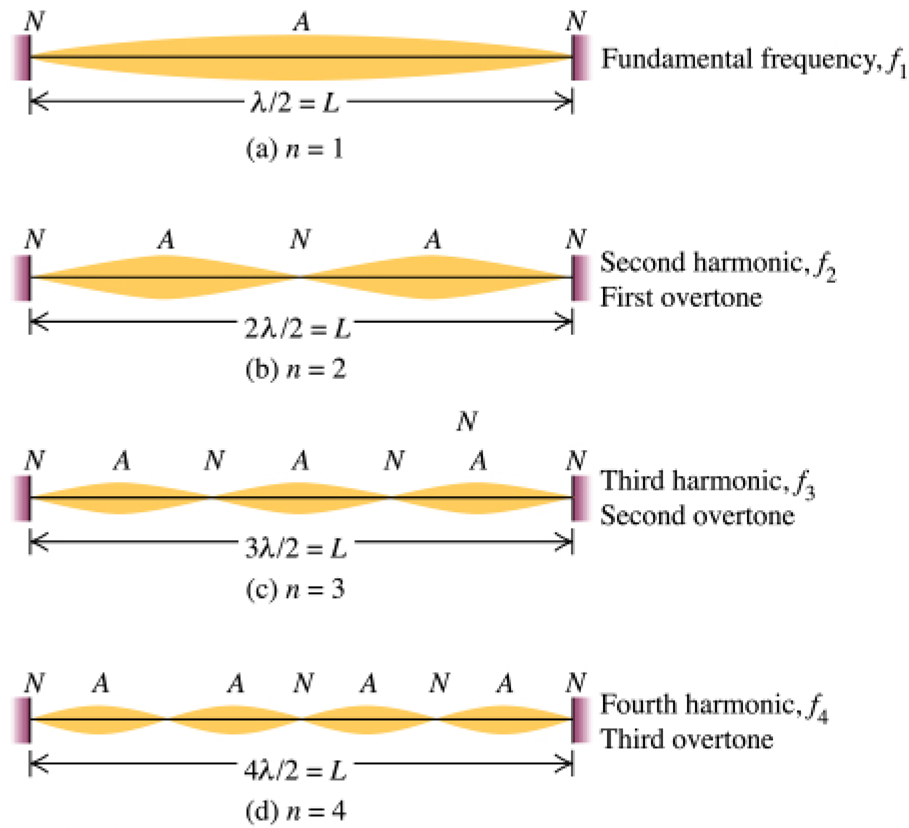
Consider a string (length L, tension F and mass/length μ). Each normal mode of the vibrating string (standing wave) is labeled by an integer n and has a characteristic wavenumber k_n, vibration frequency f_n and wave function $y_n(x,t)$

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}; \text{ using } \lambda_n = \frac{2L}{n}$$

$$f_n = \frac{\omega_n}{2\pi} = nf_1 = \frac{nv}{2L}; \quad \text{where } v = \sqrt{\frac{F}{\mu}}$$

$$y_n(x,t) = A\sin(k_n x)\cos(2\pi f_n t)$$

$$y_n(x,t) = A\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi v}{L}t\right)$$

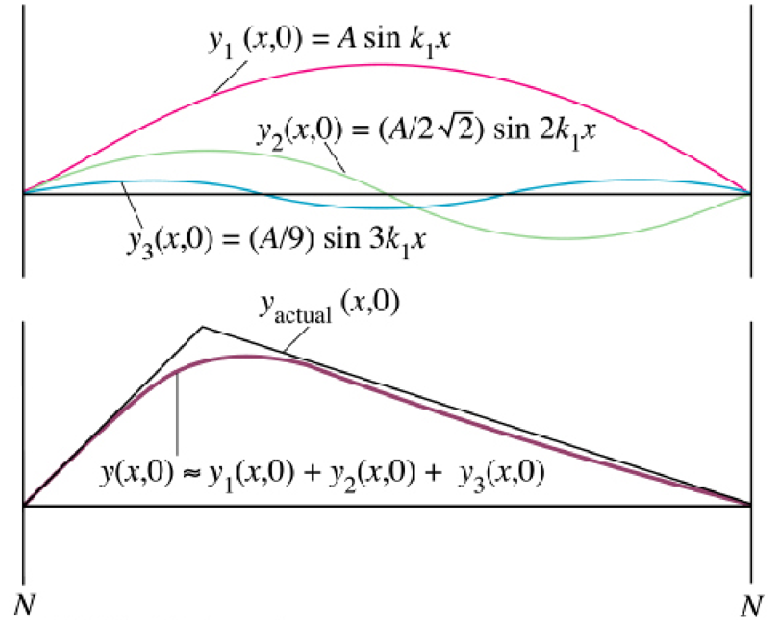


Note any linear combination (superposition) of normal mode wave functions also satisfies the wave equation and boundary conditions and is therefore a possible mode of vibration. However normal modes are the simplest type of vibration or wave function since all parts of the string vibrate at a single frequency.

Superposition of Normal Modes

Now suppose the string is plucked at t=0 (i.e. pulled at one point x_0 and then released). Then at t=0 the wave function looks like a triangle [$y_{actual}(x,0)$].
One can approximate $y_{actual}(x,0)$ as a linear combination of the first three normal mode wave functions evaluated at t=0.

What is the time dependence of the string at x=L/2?



$$y(x,t) = A[\sin(k_1 x)\cos(k_1 vt) + 2\sqrt{2}\sin(2k_1 x)\cos(2k_1 vt) + \frac{1}{9}\sin(3k_1 x)\cos(3k_1 vt)]$$

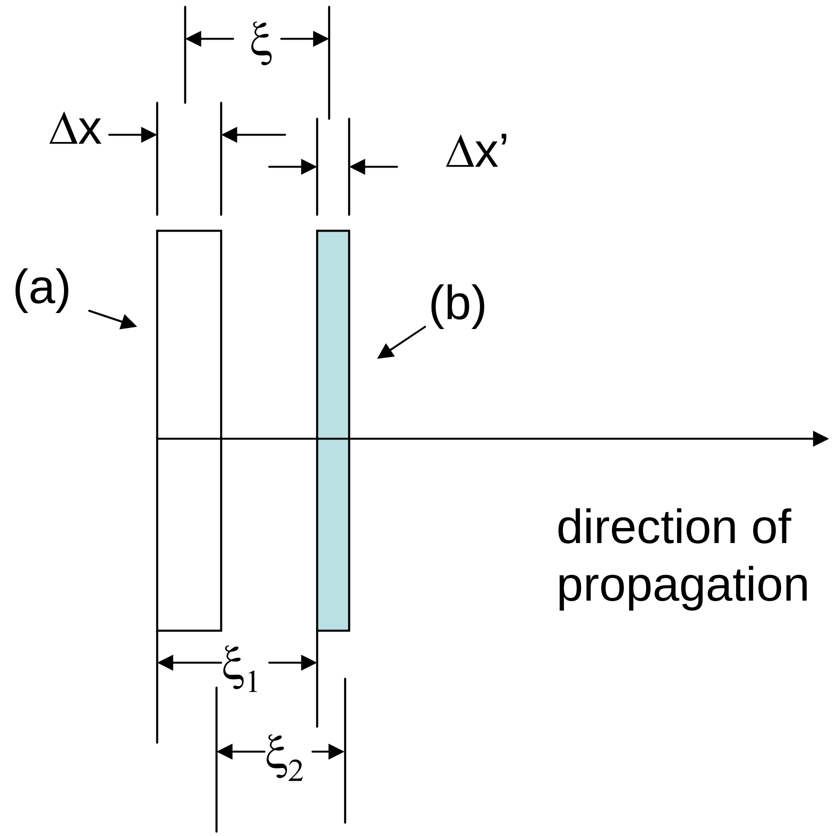
$$y(\frac{L}{2},t) = A[\sin\left(\frac{\pi}{L}x\right)\cos\left(\frac{\pi}{L}vt\right) + 2\sqrt{2}\sin\left(\frac{2\pi}{L}x\right)\cos\left(\frac{2\pi}{L}vt\right) + \frac{1}{9}\sin\left(\frac{3\pi}{L}x\right)\cos\left(\frac{3\pi}{L}vt\right)]$$

$$= A\left[\cos\left(\frac{\pi}{L}vt\right) - \frac{1}{9}\cos\left(\frac{3\pi}{L}vt\right)\right]$$

Note in general the string may vibrate at the fundamental frequency f_1 plus all the higher harmonics which are multiples of the fundamental. The amplitude of the vibration at each frequency will depends on how the string is plucked (initial position) and where on the string you measure the vibration.

Wave Equation and Velocity for a Compressional Wave

In a compressional wave(e.g. sound) the displacement is in the direction of wave propagation. Consider a thin layer of material with density ρ of area A in equilibrium (slab a) . As the wave passes the back face of the slab is displaced by ξ_1 and the front face by ξ_2. The displaced slab is as shown in (b).



The net force on slab at (b) is equal to the pressure difference (Δp) on both sides of the slab times area A. Applying F=ma to the slab with center of mass displacement ξ:

$$\underset{F}{-\Delta p A} = \underset{m}{(\rho A \Delta x')}\underset{a}{\frac{d^2\xi}{dt^2}} \text{ or } \frac{dp}{dx} = -\rho\frac{d^2\xi}{dt^2};\ \text{Eqn 1}$$

Now use the bulk modulus (B) to evaluate LHS

$$p = B\frac{\Delta V}{V} = B\frac{-A(\xi_2-\xi_1)}{A\Delta x} = -B\frac{\Delta\xi}{\Delta x} = -B\frac{d\xi}{dx}$$

$$\frac{dp}{dx} = -B\frac{d^2\xi}{dx^2};\ \text{Eqn 2}$$

Inserting Eqn2 into Eqn 1 gives $\displaystyle\frac{d^2\xi}{dx^2} = \frac{\rho}{B}\frac{d^2\xi}{dt^2}$; compare with wave equation for a string

what is the wave velocity = ??

Wave velocity in an ideal gas

Recall $PV^{\gamma} = \alpha(\text{constant})$ in and ideal gas for an adiabatic process

$\text{P} = \alpha V^{-\gamma}$; where $\gamma = \text{C}_\text{p} / C_v$

$$\frac{dP}{dV} = -\gamma\alpha V^{-\gamma-1} = -\gamma PV^{\gamma} \cdot V^{-\gamma-1} = -\gamma PV^{-1}$$

$$\frac{dP}{dV/V} = -\gamma P \Rightarrow B = \gamma P$$

but LHS = −bulk modulus B and therefore the velocity

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\gamma P}{\rho}}$$

substitute in $\rho = \dfrac{m}{V} = \dfrac{nM}{V} = M\dfrac{P}{RT}$; using $PV = nRT$

$v = \sqrt{\dfrac{\gamma RT}{M}}$; where M is the molar mass. Note independent of pressure!

Example: What is the velocity of sound in air 300K ?