Noncommutative Geometry, Matrices and String Theory

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Abstract

This thesis is a study of several aspects of noncommutative geometry arising in the context of string theory and field theories. The first chapter contains a brief introduction to noncommutative geometry, surveying relevant topics in string theory and exploring the connections between noncommutative geometry and string theory. Four different topics are then studied in depth: the noncommutative model of the D1-D3 brane intersection in a background magnetic field, intersecting solitons and their fluctuations in noncommutative scalar field theory, gravity duals of nonlocal field theories, and the M(atrix) model for the noncommutative deformation of the (2,0) theory on the five-brane.

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To my parents

Contents

Chapter 1

Introduction

In this introductory chapter, we review basic concepts from several recent topics in string theory central to the material in this thesis. These topics are: noncommutative geometry, nonabelian coordinates of D-branes, Matrix models, and noncommutative geometry in string theory. The subject is vast, and we treat all topics in the most cursory fashion. The main purpose of this chapter is to show how noncommutative geometry arises from string theory, and how its many different manifestations are connected.

The last section of this chapter contains an outline of the rest of the thesis.

For clarity of nomenclature, the word *noncommutative* will always refer to a property of spacetime coordinates, while the word *nonabelian* will refer to a property of gauge groups. Hence, a noncommutative gauge theory is a gauge theory defined on a noncommutative space, and a nonabelian gauge theory is a gauge theory with a nonabelian gauge group. The two properties being independent, one can speak of nonabelian commutative gauge theories, abelian noncommutative gauge theories and nonabelian noncommutative gauge theories.

1.1 What is noncommutative geometry?

Noncommutative geometry in simplest terms is the idea that space coordinates (say, x and y) do not have to commute with each other, *i.e.*, that $xy - yx = [x, y] \neq 0$. Since commutation relations are used in quantum mechanics to express uncertainty, noncommuting coordinates were proposed to quantize space at small distances. Until recently, such proposals were not taken very seriously, since they require an *a priori* violation of locality and Lorentz invariance. The idea of noncommutative geometry was first put on a solid mathematical footing by Alain Connes [33] in 1980, where it was applied to noncommutative tori. In 1997 noncommutative gauge theories were realized as limits of M-theory and string theory [34, 44], raising an interest in noncommutative theories of all kinds. Since then, noncommutative geometry has been shown to appear in string theory in several different guises. It provides a testing ground for the consequences of nonlocality in a setting simpler than that of a full blown string theory. Using noncommutative theories as an effective description of certain string scenarios has led to new insights into the physics of these scenarios. Outside of string theory, noncommutative deformations of well known theories (such as the Yang-Mills theory) provide an insight into these theories by introducing an extra parameter in which to do perturbation theory. Sometimes (as, for example, in chapter 5 of this thesis), introducing noncommutativity resolves certain singularities, or regulates the theory under study.

Noncommuting coordinates have been introduced in theoretical condensed matter physics, as a formalism to describe the physics of the Quantum Hall Effect (QHE) at the first Landau level [57]. Since this is a simple example of noncommutativity, arising in a physically significant situation, we will start with a review of this set-up (see, for example, [56]).

The QHE is a phenomenon exhibited by a two-dimensional electron gas in a strong uniform magnetic field normal to the plane of the system, characterized by a nearly vanishing dissipation and by quantization of the Hall conductance in units of e^2/h (for more details, see [56]). The Hamiltonian for this system, in symmetric gauge

$$A = -\frac{1}{2} r \times B = (-By/2, Bx/2) \tag{1.1}$$

is

$$H_{QHE} = \frac{1}{2m_e}\left(p + \frac{e}{c}A\right)^2 = \frac{1}{2m_e}p^2 + \frac{e}{2m_e c}BL + \frac{e^2B^2}{8m_e c^2}(x^2+y^2) \ , \tag{1.2}$$

where $L = xp_y - yp_x$ is the angular momentum. The spectrum of this Hamiltonian contains highly degenerate Landau levels with energy $(n+\frac{1}{2})\hbar\omega_c$, where $\omega_c = eB/m_e c$ is the cyclotron frequency. At the lowest Landau level, $\frac{1}{2}\hbar\omega_c$, the solutions are parameterized by a nonnegative integer m, corresponding to the angular momentum: $L \cdot \phi_m = -\hbar m \phi_m$ (the awkward minus sign is necessary to conform to high-energy conventions about which functions are considered holomorphic and which are considered antiholomorphic), and can be explicitly written in terms of $z \equiv (x+iy)/l$ as

$$\phi_m = \frac{1}{\sqrt{2\pi l^2 2^m m!}} \bar{z}^m e^{-\frac{1}{4}|z|^2} \ , \tag{1.3}$$

where

$$l \equiv \sqrt{\frac{\hbar c}{eB}} \ . \tag{1.4}$$

Since *any* nonnegative m is allowed, this implies that any wavefunction of the form

$$f(\bar{z}) e^{-\frac{1}{4}|z|^2} \tag{1.5}$$

(for $f(\bar{z})$ an antiholomorphic function of z) is a solution to Hamilton's equation with energy $\frac{1}{2}\hbar\omega_c$. Hence, the subspace of the lowest Landau level (LLL) is equivalent to the Hilbert space of antiholomorphic functions, on which we can define an inner product

$$\langle g, f\rangle \equiv \int \frac{dx\,dy}{2\pi} g^* f e^{-\frac{1}{2}|z|^2} \ . \tag{1.6}$$

Under this equivalence

$$\phi(z) = f(\bar{z})e^{-\frac{1}{4}|z|^2} \in \mathrm{LLL} \quad \leftrightarrow \quad f(\bar{z}) \ , \tag{1.7}$$

the action of $\hat{z}^\dagger$ on any $\phi \in$ LLL corresponds to multiplication by $\bar{z}$ of $f(\bar{z})$,

$$\hat{z}^\dagger \cdot \phi \quad \leftrightarrow \quad \bar{z} f(\bar{z}) \ . \tag{1.8}$$

The adjoint of this operator, though, when restricted to LLL, is not the multiplication by z, but corresponds to the action of $2\partial/\partial\bar{z}$ on $f(\bar{z})$, as can be seen from

$$\begin{aligned} \langle \bar{z} g, f\rangle = \langle g, z f\rangle &= \int \frac{dx\,dy}{2\pi} g^* f \left(-2\frac{\partial}{\partial \bar{z}}\right) e^{-\frac{1}{2}|z|^2} \\ &= \int \frac{dx\,dy}{2\pi} g^* \left(2\frac{\partial}{\partial \bar{z}} f\right) e^{-\frac{1}{2}|z|^2} = \langle g, 2\frac{\partial}{\partial \bar{z}} f\rangle \ , \end{aligned} \tag{1.9}$$

valid for all g. Thus,

$$\hat{z} \cdot \phi \quad \leftrightarrow \quad 2\partial/\partial\bar{z} f(\bar{z}) \ . \tag{1.10}$$

We obtain the following commutation relationship, valid when the physics is restricted to the LLL:

$$[\hat{z}, \hat{z}^\dagger] = 2 \ . \tag{1.11}$$

Recalling that $z = (x + iy)/l$ and defining $\theta \equiv l^2$ we obtain

$$[\hat{x}, \hat{y}] = i\theta = i\frac{\hbar c}{eB} . \tag{1.12}$$

This equation captures the essence of noncommutative geometry. By restricting our attention to the lowest Landau level, we have created a one-particle quantum mechanical system in which the coordinates do not commute with each other.

Before we talk any further about the properties of this system, let us generalize to an arbitrary number of noncommutative directions d. The equation above generalizes to

$$\boxed{[\hat{x}^i, \hat{x}^j] = i\theta^{ij}} \tag{1.13}$$

where i, j run from 1 to d and θ is a $d \times d$ antisymmetric matrix. For simplicity, we will assume here that θ has full rank, which means maximal noncommutativity. This necessarily implies that d is even.

In order to study this further, we define an algebra of all complex linear combinations of products of the variables $\hat{x}^i$, denoted by $\mathbb{R}^d_\theta$. Since different $\hat{x}^i$s do not commute, this is a noncommutative algebra. We will now briefly review the properties of this algebra, and the associated geometry, roughly following [47].

In order to talk about the physics in this set-up, we will need concepts of derivatives and integrals. A simple set of derivatives can be defined by the relations

$$\partial_i \hat{x}^j = \delta_i^j \tag{1.14}$$

$$[\partial_i, \partial_j] = 0 . \tag{1.15}$$

These relations allow us to calculate the derivative of any element of $\mathbb{R}^d_\theta$, using $\partial_i(\hat{f}\hat{g}) = \partial_i(\hat{f})\hat{g} + \hat{f}\partial_i(\hat{g})$ for any $\hat{f}, \hat{g} \in \mathbb{R}^d_\theta$. The integral is fixed (up to a normalization), by the requirement that $\int \partial \hat{f} = 0$. To see this, consider the momentum basis for $\mathbb{R}^d_\theta$, which is defined as the set of eigenfunctions of the derivative operators

$$\partial_i e^{ik\hat{x}} = ik_i e^{ik\hat{x}} \ . \tag{1.16}$$

In this basis, the integral is simply defined as

$$\int e^{ik\hat{x}} = \delta(k) \ . \tag{1.17}$$

This abstract definition of $\mathbb{R}^d_\theta$ seems to have little to do with geometry. We would like to be able to think of $\mathbb{R}^d_\theta$ as a deformation of $\mathbb{R}^d$. In order to make a connection between these two objects, we define a linear map from $\mathbb{R}^d_\theta$ to functions on $\mathbb{R}^d$:

$$S: \ \hat{f} \ \rightarrow \ f(x^i) \ , \tag{1.18}$$

such that

$$S[\hat{f}\hat{g}] = S[\hat{f}] \star S[\hat{g}] \ , \tag{1.19}$$

where $\star$ is some deformed law of multiplication for functions on $\mathbb{R}^d$, different from the usual pointwise multiplication.

The inversion of S poses an ordering ambiguity. What should $S^{-1}(x_1 x_2)$ be? It could be $\hat{x}_1\hat{x}_2$, or $\hat{x}_2\hat{x}_1$, or maybe $\frac{1}{2}(\hat{x}_1\hat{x}_2 + \hat{x}_2\hat{x}_1)$. The most common choice for S^{-1} is the Weyl ordering, defined by

$$\hat{f}(\hat{x}) \ = S^{-1}[f] \ = \frac{1}{(2\pi)^d}\int d^dk \ e^{ik\hat{x}} f(k) \ , \tag{1.20}$$

$$f(k) \ = S[\hat{f}](k) \ = \int e^{-ik\hat{x}} \hat{f}(\hat{x}) \ . \tag{1.21}$$

From this definition, we can compute the star-product, $\star$. For the basis elements in momentum space, we easily get that

$$e^{ikx} \star e^{ik'x} = e^{-\frac{i}{2}\theta^{ij}k_i k'_j}\ e^{i(k+k')x} \tag{1.22}$$

using the CBH formula. From this, we can compute the star-product in position space

$$\begin{aligned}
f(x) &= \frac{1}{(2\pi)^d}\int d^dk\ f(k)e^{ikx} \\
g(x) &= \frac{1}{(2\pi)^d}\int d^dk'\ g(k')e^{ik'x} \\
(f\star g)(x) &= \frac{1}{(2\pi)^d}\int d^dk\ \frac{1}{(2\pi)^d}\int d^dk'\ f(k)g(k')e^{ikx}\star e^{ik'x} \\
&= \frac{1}{(2\pi)^d}\int d^dk\ f(k)e^{ikx}\frac{1}{(2\pi)^d}\int d^dk'\ g(k')e^{ik'x'}\ e^{-\frac{i}{2}\theta kk'}\Big|_{x=x'} \\
&= \exp\left(\frac{i}{2}\theta^{ij}\partial_{x_i}\partial_{x'_j}\right) f(x)g(x')\Big|_{x=x'} .
\end{aligned}$$

This is known as the Moyal star-product:

$$\boxed{(f\star g)(x) = \exp\left(\tfrac{i}{2}\theta^{ij}\partial_{x_i}\partial_{x'_j}\right) f(x)g(x')\Big|_{x=x'}} \tag{1.23}$$

We now have two different ways to think about noncommutative geometry. We can either think in terms of a noncommutative algebra of operators $\hat{x}^i$, or in terms of functions of commutative variables x^i whose multiplication is defined to be the star-product, $\star$.

The noncommutative algebra picture becomes more clear when we interpret the elements of the algebra as acting on a Hilbert space of the harmonic oscillator (HO). Restricting the discussion to two dimensions, $d = 2$, and with positive θ defined by

$[\hat{x}^1, \hat{x}^2] = i\theta$, we can define raising and lowering operators

$$a \equiv (\hat{x}^1 + i\hat{x}^2)/\sqrt{2\theta} \tag{1.24}$$

$$a^\dagger \equiv (\hat{x}^1 - i\hat{x}^2)/\sqrt{2\theta} \tag{1.25}$$

with standard commutation relations

$$[a, a^\dagger] = 1 \ . \tag{1.26}$$

Thus, any product of $\hat{x}^i$s becomes a Fock space operator acting on the Hilbert space of a HO. The integral over all $\mathbb{R}^d_\theta$ becomes the trace over the Hilbert space:

$$\int \hat{f} = (2\pi\theta)\ Tr_{HO} \ , \tag{1.27}$$

where the normalization can be checked using the definition (1.17).

The Fock space formalism can be extended to an arbitrary even number of dimensions d as follows. By applying a linear transformation to the coordinates $\hat{x}^i$, we can bring the antisymmetric matrix θ^{ij} into canonical form. The new coordinates are then pairwise noncommutative, and the problem breaks into $d/2$ independent HOs. Thus, the noncommutative algebra can be interpreted as the Fock space of $d/2$ independent HOs.

We have treated the flat non-compact noncommutative space in some detail. Noncommutative algebras can also be constructed for other geometries. Let us briefly examine two of the more important ones - the noncommutative torus and the noncommutative sphere.

The noncommutative 2-torus is defined using two matrices U and V such that

$$UV = \alpha^{-1} VU \ , \tag{1.28}$$

for $\alpha = e^{2\pi i/N}$. If $U = e^{i\hat{x}}$ and $V = e^{i\hat{y}}$ with $[\hat{x}, \hat{y}] = i\theta$, then we would have $\theta = 2\pi/N$. However, we don't have to define U and V in terms of $\hat{x}$ and $\hat{y}$; instead, we define them in terms of equations (1.28). The advantage to this is that relation (1.28) can be realized in a finite representation, as the 't Hooft clock-and-shift N×N matrices

$$\begin{aligned} U &= \begin{pmatrix} 1 & & & & \\ & \alpha & & & \\ & & \alpha^2 & & \\ & & & \ddots & \\ & & & & \alpha^{N-1} \end{pmatrix} \\ V &= \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{pmatrix} \end{aligned} \tag{1.29}$$

We then interpret this as a 2-torus $(x, y) \sim (x + 2\pi, y) \sim (x, y + 2\pi)$, by identifying U with e^{ix} and V with e^{iy}. This can easily be extended to d dimensions by taking $U_i U_j = e^{-i\theta_{ij}} U_j U_i$. In order to replace any function on the d-torus with an element in the noncommutative algebra, we write the function as a Fourier sum and replace

e^{ix_i} with U_i:

$$f(x_1, \ldots, x_d) = \sum_{n_i, i=1,\ldots,d} c_{\mathbf{n}} e^{in\cdot x} \;\rightarrow\; \sum_{n_i, i=1,\ldots,d} c_{\mathbf{n}} \prod_j U_j^{n_j} \; . \tag{1.30}$$

The noncommutative torus was the first extensively studied noncommutative system [33, 34, 44]. It is central to the construction of the M-theory membrane in Matrix theory, as will be explained in section 1.3.

Another noncommutative algebra with a classical geometric interpretation and a finite matrix representation is the noncommutative sphere $\mathbf{S}^2_\theta$. Consider the algebra generated by 3 N×N matrices x^i for $i = 1, 2, 3$, with the commutation relationship

$$[x^i, x^j] = i\frac{2R}{N}\epsilon^{ijk}x^k \; . \tag{1.31}$$

This is a (rescaled) N-dimensional representation of the SU(2) algebra. The classical interpretation for large N is that of a sphere with radius R + $\mathcal{O}(1/N^2)$,

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = R^2 \left(1 - \frac{1}{N^2}\right) \text{ Id} \; . \tag{1.32}$$

This construction is also used in the Matrix model of M-theory, to construct compact membranes. We will have more to say about the noncommutative sphere in chapter 2.

1.1.1 Noncommutative gauge field theories

In this subsection, we will briefly introduce the basic concepts in noncommutative gauge field theories. For more details, see [47, 79, 80].

As always, the starting point is a gauge field A, which here takes values in the noncommutative algebra. We define the associated field strength to be

$$F_{ij} \equiv \partial_i A_j - \partial_j A_i + i[A_i, A_j] \ . \tag{1.33}$$

The important thing to keep in mind is that the commutator does not vanish even for an abelian theory, since A is an element of a noncommutative algebra.

A transforms under gauge transformations with an infinitesimal parameter ϵ as

$$\delta_\epsilon A_i = \partial_i \epsilon + i[A_i, \epsilon] \ , \tag{1.34}$$

which implies that the field strength transforms as

$$\delta_\epsilon F_{ij} = i[F_{ij}, \epsilon] \ , \tag{1.35}$$

and thus we have a gauge invariant action

$$S = -\frac{1}{4g^2} \int F^2 \ . \tag{1.36}$$

The interesting thing is that the gauge symmetry group of this theory is much larger that in the commutative case. For example, it includes spacetime translations. Let $\epsilon = u^i (\theta^{-1})_{ij} x^j$. Then

$$\delta_\epsilon A_i = u^j \partial_j A_i + u^j (\theta^{-1})_{ji} \ . \tag{1.37}$$

Ignoring the physically irrelevant constant shift in A, this is nothing but a translation by u^i. In fact, the gauge symmetry group contains all spacetime diffeomorphisms.

Taking the unification of spacetime symmetries and gauge symmetries further, notice that we can replace the covariant derivatives $D_i = \partial_i + iA_i$ with an operator

$$C_i \equiv (-i\theta^{-1})_{ij} x^j + iA_i \tag{1.38}$$

so that

$$D_i \to [C_i, \cdot] \ . \tag{1.39}$$

Then, the field strength is just

$$F_{ij} \to i[C_i, C_j] - (\theta^{-1})_{ij} \ , \tag{1.40}$$

and the Yang-Mills action becomes the 'Matrix model' action

$$S = Tr\left([C_i, C_j] - (\theta^{-1})_{ij}\right)^2 \ . \tag{1.41}$$

This action for matrix variables C_i arises in many different contexts. How is the d-dimensional (and noncommutative) space to emerge back from this action? The answer lies in the $(\theta^{-1})_{ij}$-term. This term limits the configurations under consideration to those with $Tr([C_i, C_j]) = (\theta^{-1})_{ij}$. We can think of each θ as defining a topologically different sector of the matrix model. Fixing this topological invariance fixes both the dimensionality of the noncommutative space and the noncommutativity parameter θ. We will see that a nontrivial trace for the elements of the matrix algebra under consideration is a common theme in the ways in which noncommutative geometry arises in string theory and in matrix models.

Having introduced the basic concepts in noncommutative geometry, we can now proceed to explore how it arises naturally in the string theoretic setting.

1.2 Nonabelian branes

In its perturbative formulation, closed string theory is a two dimensional sigma model, the target space of which is identified with spacetime. The various fields living in spacetime, such as the metric and the different gauge fields, form the parameters of the sigma model. Their equations of motion are determined by requiring that the sigma model be conformally invariant on worldsheets of any genus (higher genii corresponding to string loop corrections). Thus, the equations of motion for all background fields are computed by setting the beta functions of the corresponding couplings to zero.

The introduction of open strings is equivalent to introducing a boundary in the sigma model worldsheet. Different boundary conditions correspond to objects in the target space-time. These objects can roughly be called the 'branes' in the theory. A brane is nothing more than that part of the background which determines the boundary conditions for the propagation of an open string. Again, the boundary conditions must be such as to define a 'good' boundary conformal field theory (BCFT) on the worldsheet. What constitutes a 'good' BCFT is still an open problem (see [51] for a review and a list of further references).

The most studied and best understood branes in string theory are the Dp-branes (for an introduction, see for example [92, 93, 96, 95, 111]). A Dp-brane is a (p+1)-dimensional surface in the target spacetime on which open strings can end. Their importance was first appreciated when Polchinski pointed out [94] that they correspond to certain solitonic solutions in supergravity and are charged under the RR form-fields. Thus, the D-branes are nonperturbative objects in string theory, the study of which opened a window into previously inaccessible regimes.

A single Dp-brane in (D+1) dimensions is described by its position and a worldvolume U(1) gauge field A of dimension (p+1). Because of diffeomorphism invariance, the position can be described by (D-p) scalar fields Φ, corresponding to transverse displacements from some fixed configuration. Together, there are (D+1) bosonic degrees of freedom, corresponding to the (D+1) massless bosonic string modes for an open string with its ends on the Dp-brane. In addition, for a superstring, there are appropriate fermionic fields, which we will ignore. The action for the background fields A and Φ can be determined as described above, by requiring conformal invariance of the BCFT. In a purely bosonic theory, the answer is the Born-Infeld action [82],

$$S_{BI} = -T_p \int d^{p+1}\xi \; e^{-\phi} \sqrt{-det\,(P[G+B] + 2\pi\alpha' F)}\;, \tag{1.42}$$

where T_p is the brane tension, $P[G+B]$ denotes the pullback of the background metric and the NSNS two-form field to the worldvolume of the brane, ϕ is the dilaton and $F = dA$ is the field strength corresponding to the gauge potential A. For a flat background metric $G = \eta$, with $B = 0$ and $\phi =$ const, a flat Dp-brane localized at $X^i = 0$ for $i = p+1, \ldots, D+1$ in static gauge $X^i = \xi^i$ for $i = 0, \ldots, p$, with small transverse displacements Φ, and a weak gauge field A, the Born-Infeld action simplifies to the U(1) Yang-Mills action (plus a constant corresponding to the energy density of the Dp-brane, which we will ignore):

$$S = -\frac{1}{4g_{YM}^2} \int d^{d+1}\xi \;\left(F^2 + \frac{2}{(2\pi\alpha')^2} (\partial X)^2 \right) + \text{fermions}\;. \tag{1.43}$$

The scalars Φ transform in the adjoint of the U(1) (*i.e.* trivially).

Consider now N flat coincident Dp-branes in this same setting. As was first pointed out by Witten [119], there are now N^2 massless open string modes, since each string has two ends that can be attached to either one of the N branes. It turns out that the massless fields arrange themselves precisely in the right way to fill out the gauge field components and the adjoint scalars of a U(N) YM theory. The low energy description of N flat coincident Dp-branes in a trivial background is thus the U(N) gauge theory with (D-p) scalars corresponding to the transverse displacements. The nonabelian generalization of the Born-Infeld action, which would describe N coincident Dp-branes in arbitrary (but slow-varying) backgrounds to all orders in α' is not fully known. There are proposals for this action, see [88, 115] and references therein for more details. We will talk more about the nonabelian Born-Infeld action and its solutions in chapter 2.

It is somewhat strange that the transverse displacements of the D-branes are now described by $N \times N$ hermitian matrices. How do we interpret these in terms of classical positions?

The answers comes when we analyze the YM action. It contains a potential term proportional to

$$- \, Tr \, [\Phi^i, \Phi^j][\Phi^i, \Phi^j] \; . \tag{1.44}$$

This term is minimized when the matrices Φ^i all commute with each other. In that case, they can be simultaneously diagonalized, and their eigenvalues correspond to the classical positions of the N D-branes. The residual S_N permutation symmetry corresponds to permuting the (identical) branes.

The question then arises: What if the matrix coordinates of the D-branes cannot be simultaneously diagonalized? In this case, there is no classical interpretation of the coordinates and the geometry arising in such a set up is intrinsically noncommutative. We will see shortly how essentially all known instances of noncommutative geometry in string theory arise precisely from this matrix nature of D-brane geometry.

The simplest, and at the same time most generic, case of noncommutative geometry in string theory arises from a further analysis of (1.44) (see [103] for a review and references therein for further details). For a static configuration, the equations of motion for Φs are simply

$$[\Phi^i, [\Phi^i, \Phi^j]] = 0 \ , \tag{1.45}$$

and their solutions are

$$[\Phi^i, \Phi^j] = \theta^{ij} \ \mathrm{Id}_{N\times N} \ . \tag{1.46}$$

This is precisely the commutation relationship for the generating coordinates in noncommutative geometry (1.13)! There is just one problem - the equation above can only be satisfied with N infinite. Let us for a moment accept the possibility of an infinite number of Dp-branes. If the rank of θ is r, we can then interpret (1.46) as a configuration in which r of the transverse directions have 'expanded' to form a D(p+r)-brane with r noncommutative directions. It can then be shown (see, for example, [103]), that the effective action for this object is the YM action in (p+r) dimensions, r of which are noncommutative. We will return to this point later, when discussing matrix models.

The above simple example shows that it is possible to build higher dimensional D-branes out of lower dimensional ones (we will see more examples of that, including cases involving a finite number of D-branes, in Chapter 2). Is it also possible to obtain lower dimensional branes from higher dimensional ones? The answer is *yes*. In fact, it is *yes* in more than one way. Here, we will discuss how dissolved lower dimensional branes can be thought of as being equivalent to worldvolume gauge fields in a higher dimensional brane. In chapter 3, we will talk about another way of obtaining lower dimensional branes from higher dimensional ones, as solitons in the string tachyon field.

The Born-Infeld action for a single D-brane is only half the story. As was mentioned above, a Dp-brane is also a source for the corresponding RR form-field. For example, a D2-brane in type IIA string theory is a source for the three-form field $A^{(3)}$. But that is not its only coupling to the RR sector fields. The total coupling of a single Dp-brane to the RR fields can be compactly written as a Chern-Simon action term

$$S_{CS} = \mu_p \int d^{(p+1)}\xi \; \left(P \left[\sum_k A^{(k)} \right] \right) \wedge e^{2\pi\alpha' (F+B)} \; . \tag{1.47}$$

The way to read this formula is to expand the exponential and retain all RR fields $A^{(k)}$ which lead to the correct total dimension of the form under the integral. For example, for a flat D2-brane in flat background and $B = 0$, we have

$$S_{CS}^{p=2,B=0} = \mu_2 \int d^{(3)}\xi \; \left(A^{(3)} + 2\pi\alpha' A^{(1)} \wedge F \right) \; . \tag{1.48}$$

So, in this case, if $\int F \neq 0$, the D2-brane carries a D0-brane charge, since it acts as a source for $A^{(1)}$. This observation was first made by Witten [120], for Dp-branes carrying D(p-4)-brane charge due to a non vanishing instanton number $\int F \wedge F$. It was then extended to all dimensions by Douglas [43]. While the argument for the Chern-Simon action above can be made from string theory using anomaly cancellation, simpler arguments can be given from T-duality (see [111] for a review and further references).

The important point is that lower dimensional branes can be thought of as arising from nontrivial gauge field configurations on higher dimensional branes. We can think of the worldvolume gauge field as corresponding to a dissolved density of some lower dimensional D-branes.

The nonabelian version of the Chern-Simon action [88] elucidates further the dual picture of higher dimensional branes arising from lower dimensional ones. This action is

$$S_{CS} = \mu_p \int STr \left(P \left[e^{2\pi i \alpha' i_\Phi i_\Phi} (\Sigma A^{(n)} e^{2\pi\alpha' B}) \right] e^{2\pi\alpha' F} \right) \; , \tag{1.49}$$

where i_Φ is an interior product by Φ^i, for example $i_\Phi i_\Phi A^{(2)} = \frac{1}{2}A^{(2)}_{ij}[\Phi^i, \Phi^j]$. Since each i_Φ 'eats' an index, it reduces the dimension of the form and allows a coupling to higher dimensional RR fields. As an example, D0-branes couple to $A^{(3)}$

$$\mu_p \int STr\left(P\left[\frac{1}{2}A^{(3)}_{ijk}(2\pi\alpha')[\Phi^i, \Phi^j]\right]\right) . \tag{1.50}$$

For concreteness, let $[\Phi^1, \Phi^2] = \theta$. Then, we can think of having one D0-brane per cell with an area θ (up to some numerical constants) and hence θ^{-1} as the density of D0-branes forming a D2-brane. This density of D0-branes should correspond to a gauge field on the resulting D2-brane equal to $F = \theta^{-1}$ and it does - see chapter 2 for the details. The point to take away is this: bound states of D-branes of different dimensions can be described from two different points of view, that of the higher dimensional branes and that of the lower dimensional branes. When described in terms of the higher dimensional branes, the lower dimensional ones arise as nontrivial gauge field configurations. When described in terms of the lower dimensional branes, the higher dimensional ones arise as nontrivial commutator terms for transverse coordinates.

The discussion above was meant to convey the general ideas only. The details are a little more complicated. When discussing the non compact case, care has to be taken to deal properly with the fact that we need an infinite number of D-branes and that all worldvolume integrals are over a non compact space. This difficulty can be avoided if a compact space is used (for example, a torus) on which the branes are wrapped. By treating the situation on a torus, infinite matrices can be obtained even from a finite number of D-branes (see [111] for a review and further references). Since this set-up will not arise in the rest of this thesis, we will not talk about it here.

We started this section by defining a brane to be any consistent boundary condition for open strings. Studying these in all generality is very hard, and we quickly specialized to the restricted class of D-brane boundary conditions, defined as a restriction to some hyperplane in the target spacetime. Yet, from the above discussion, it is quite clear that to say that a 'D-brane is a surface on which strings can end' is misleading. Nonabelian brane configurations can lead to physically complicated systems with little or no geometrical interpretation, and hence to boundary conditions we would originally not recognize as D-branes. Studying noncommutative geometry as it arises in nonabelian brane configurations is a way of understanding the physics behind at least some of the configurations with no classical interpretation in geometry, leading perhaps to some understanding of all branes.

The fact that D-branes of one dimension can be constructed out of D-branes of another dimension points perhaps to the existence of some 'stuff' out of which all D-branes are 'made'. This 'stuff' would be a good candidate for the degrees of freedom in a nonperturbative formulation of string theory. The Matrix model of M-theory, which is our next subject, suggests that this 'stuff' is simply D0-branes.

1.3 M(atrix) models

In this section, we will briefly review some key ideas of the Matrix model. For more details, see the reviews [13, 14, 111, 114].

The Matrix model for M-theory was proposed in 1997 by Banks, Fischler, Shenker and Susskind (BFSS) [15]. They were motivated by the work on D0-branes and by string dualities. When M-theory is compactified on a small space-like circle, the resulting theory is the type IIA string theory, and the quanta corresponding to momentum in the compact direction are the D0-branes. In a frame of reference with a very large momentum around the compact direction, the dynamics of 11-dimensional

M-theory becomes essentially non relativistic. In the dual IIA string theory, this dynamics is described by the large N limit of a non relativistic system of D0-branes. The BFSS conjecture was later extended to include finite N [110, 102], in a set-up where M-theory is compactified on a light-like circle.

As was described in the previous section, the dynamics of N D0-branes is the SYM theory in $(1+0)$ dimensions. The Lagrangian can be obtained from 10-dimensional SYM by dimensional reduction and is equal to [114]

$$\mathcal{L} = \frac{1}{2gl_s}\, Tr\, \left[\, \dot{X}^a \dot{X}^a + \frac{1}{2}[X^a, X^b]^2 + \text{fermions} \,\right] \tag{1.51}$$

in fixed gauge $A_0 = 0$. The X^as are the 9 $N \times N$ matrixes describing the positions of the D0-branes.

This Lagrangian was obtained much before BFSS, in the context of quantizing the supermembrane (see [114] for details and citations to original work). The idea there was to regularize the quantum theory of a two-dimensional membrane by approximating its classical geometry with a set of matrices, in a fashion similar to that used to obtain the noncommutative versions of the torus and the sphere in section 1.1. The matrix model for the membrane was thought to be ill behaved, since it contained a continuous spectrum due to instabilities associated with the membrane developing long, thin spikes. The BFSS model reinterpreted the theory as being the second quantized theory of the membrane, and not, as was originally assumed, the first quantized theory. This naturally explained the existence of the continuous spectrum.

How can the quantum mechanical model given by (1.51) contain a second quantized version of M-theory? The first important fact, resting on the supersymmetric nature of the action, is that the theory contains a bound state of N D0-branes with zero binding energy. This bound state is interpreted in the 11-dimensional theory as a graviton with N units of momentum in the compact direction. Thus, a single graviton multiplet in 11 dimensions with $p_{10} = N/R$ is represented by N D0-branes. The corresponding wave function is $\psi_B(X^i_{N\times N})$, and it has the property that $\psi_B(X^i_{N\times N}) \to 0$ when $X^i_{N\times N}$ is far from $r^i I_{N\times N}$, where r^i is the transverse position of the graviton. A multi-particle state can be built out of block-diagonal matrices, each block of size N_a corresponding to one graviton with momentum $p_{10} = N_a/R$. The (approximate) wavefunction for k particles is claimed to be [13]

$$\psi = \psi(\mathbf{r^1}...\mathbf{r^k}) \exp\left(-\frac{1}{2}|\mathbf{r^a} - \mathbf{r^b}|W^\dagger_{ab}W_{ab}\right) \bigotimes_{c=1}^{k} \psi_B(X^i_{N_c\times N_c}) \tag{1.52}$$

where r^a is the position of a^{th} graviton and Wab is the off-diagonal block between gravitons a and b. This wavefunction is appropriate as long as all $|\mathbf{r^a} - \mathbf{r^b}|$ are large, since then the potential energy term in the action makes the off-diagonal blocks W_{ab} into high-frequency harmonic oscillators which can be integrated out to give Gaussian ground states. This is the asymptotic state for k separated gravitons. When one or more of $|\mathbf{r^a} - \mathbf{r^b}|$ become small, the off-diagonal degrees of freedom cannot be integrated out, the different blocks interact with each other and the gravitons scatter. In the scattering process, the blocks can change size (longitudinal momentum being exchanged between gravitons), or the number of blocks might change (creation and annihilation of gravitons). In principle, the scattering cross section for any process can be obtained this way. Notice that the notion of spacetime is only recovered from the well separated asymptotic states and is meaningless when the gravitons are interacting.

Thus, in classical matrix theory the off-diagonal elements are set to zero and the gravitons do not interact. Upon quantization, the fluctuations about this configuration include the set of off-diagonal oscillators W_{ab}. These need to be integrated over to obtain the effective theory for the gravitons' motion. It can be shown that one loop effects reproduce the $1/r^7$ Newtonian interactions of 11-dimensional supergravity. Higher order loops reproduce general-relativistic corrections.

M-theory has two kinds of extended objects - the two-dimensional membrane and the five-dimensional membrane (the five-brane). The two-dimensional membrane can easily be constructed in matrix theory, using the techniques described towards the end of section 1.1. We have discussed there the toroidal and spherical case, but this analysis can be extended to any Riemann surface. As was the case with the noncommutative plane, the noncommutative deformation of any surface can be obtained by studying the lowest Landau level of electrons moving on the surface in a uniform magnetic field.

The five-brane is more difficult to study. When it is wrapped around the compact direction, it corresponds to a D4-brane in type IIA theory. Thus, its Matrix model is the large N limit of N D0-branes bound to one D4-brane [21]. When all of the five-brane is in the transverse directions, the problem is more difficult. In type IIA string we get an NS5-brane and N D0-branes, which are described by what is called 'a little string theory' [41, 101, 22]. We will have more to say about this in chapter 5.

That a large N limit of matrix quantum mechanics may contain all the physics of a theory with gravity and extended objects is quite remarkable. That there are deep connections between nonabelian matrix coordinates and noncommutative geometry should be plain by now. Noncommutative geometry allows us to study certain configurations arising from matrix mechanics in a simpler setting.

1.4 Noncommutative geometry in string theory

We have argued that noncommutative geometry arises naturally in string theory when we build higher dimensional objects from lower dimensional ones (for example, when we build higher dimensional D-branes out of D0-branes in the Matrix model). In section 1.2 we saw that the argument can be reversed, and that the lower dimensional branes can be interpreted as worldvolume fields in the higher dimensional branes. Can the arisal of noncommutative geometry be understood in this context? The answer, as given by Seiberg and Witten in [104], is *yes*. The starting point for their argument, which we will review briefly here, is a Dp-brane in a background NSNS field B_{ij}. The matrix B is assumed to have rank r, which is even. $r \leq p$, since any components of B normal to the brane can be gauged away, and since B is assumed to have no time-like components. With this field, the worldsheet action for an open string attached to the Dp-brane is

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} \partial_a x^i \partial^a x_i - \frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_{||} x^j \ , \tag{1.53}$$

where Σ denotes the string worldsheet and $\partial\Sigma$ is its boundary. This action implies the following boundary conditions along the Dp-brane

$$\left(g_{ij} \partial_{\perp} x^j + 2\pi\alpha' B_{ij} \partial_{||} x^j \right) \Big|_{\partial\Sigma} = 0 \ , \tag{1.54}$$

where g_{ij} is the background metric. With these boundary conditions, the propagator can be computed as [49, 26, 1]

$$\langle x^i(z) x^j(z') \rangle = -\alpha' \Bigg[\quad g^{ij} \log|z - z'| - g^{ij} \log|z - \bar{z}'| \tag{1.55}$$

$$+ \quad G^{ij} \log|z - \bar{z}'|^2 + \frac{1}{2\pi\alpha'} \theta^{ij} \log \frac{z - \bar{z}'}{\bar{z} - z'} \ + \ \text{const} \quad \Bigg] ,$$

where G_{ij} is the 'open string metric', given by

$$\begin{aligned} G^{ij} &\equiv \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}_S = \left(\frac{1}{g+2\pi\alpha' B}\, g\, \frac{1}{g-2\pi\alpha' B}\right) \\ G_{ij} &= g_{ij} - (2\pi\alpha')^2 \left(Bg^{-1}B\right)_{ij} \ , \end{aligned} \tag{1.56}$$

and θ^{ij} is the noncommutativity parameter,

$$\theta^{ij} = 2\pi\alpha' \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}_A = -(2\pi\alpha')^2 \left(\frac{1}{g+2\pi\alpha' B}\, B\, \frac{1}{g-2\pi\alpha' B}\right) \ . \tag{1.57}$$

$(\ )_A$ and $(\ )_S$ denote the antisymmetric and the symmetric part of a matrix, respectively. At the open string boundary, $z = \tau$ and $z' = \tau'$, for τ, τ' real, the propagator simplifies to

$$\langle x^i(\tau) x^j(\tau)\rangle = -\alpha' G^{ij} \log(\tau - \tau')^2 + \frac{i}{2}\theta^{ij}\epsilon(\tau - \tau'), \tag{1.58}$$

setting the constant in equation (1.56) to a convenient value. Also, $\epsilon(\tau) = \mathrm{sign}(\tau)$. It is clear from this expression why G_{ij} is called the open string metric - it is the effective metric as seen by the open strings in the background magnetic field. To interpret θ^{ij}, notice that at equal time, above equation gives the following commutator

$$[x^i(\tau), x^j(\tau)] = i\theta^{ij} \ . \tag{1.59}$$

Thus, x^is are noncommutative coordinates, just like in equation (1.13). In their paper, Seiberg and Witten further argue that in the $\alpha' \to 0$ limit, the effective theory for states on the brane reduces to the noncommutative deformation of the usual gauge theory which would live in the Dp-brane in zero background field. All that one has to do, they argue, to obtain the effective theory in the presence of the background B-field, is to replace the closed string metric g_{ij} with the open string metric G_{ij} and regular multiplication with the noncommutative star-product $\star$ with parameter θ in the effective action for a Dp-brane in a zero background field.

It is perhaps amusing to see how G and θ can be obtained from the nonabelian Born-Infeld theory. The nonabelian Born-Infeld action will be given in chapter 2. Here, let us just cite this action for N D0-branes in a static configuration,

$$S = -T_0 \; STr \; \sqrt{\det\left(\delta^i_j \;+\; 2\pi i\alpha'[\Phi^i,\Phi^k](G+2\pi\alpha' B)_{kj}\;.\right)} \tag{1.60}$$

Assuming that we are looking for a solution of the form $[\Phi^i, \Phi^j] = i\theta^{ij}$, the equation of motion is

$$\frac{\delta}{\delta\theta}\sqrt{\det\left(1 \;-\; 2\pi\alpha'\theta(g+2\pi\alpha' B)\right)} = 0 \;, \tag{1.61}$$

whose solution is

$$\theta = 2\pi\alpha'\left(\frac{1}{g+2\pi\alpha' B}\right)_A \quad . \tag{1.62}$$

This is the noncommutativity parameter which appeared in the Seiberg and Witten treatment. One more time, we see that the two pictures, one in which the lower dimensional object is treated as fundamental, and one in which the higher dimensional object is treated as fundamental, are complimentary and give compatible answers.

1.5 Outline of thesis

In the reminder of this thesis, we will study four different situations in which noncommutative geometry plays a role.

In chapter 2, the starting point is the nonabelian Born-Infeld action. We will study there the D3-D1 brane intersection in a nonzero NS-NS background two-form field B, as described from the point of view of the effective theory of D1-branes. This analysis follows [35], where it was shown that a bundle of D1-branes can expand out to a nonabelian 'funnel' interpreted as a D3-brane to which the D1-branes are attached. We extend this analysis to a nonzero background field, and show that not only the correct 'tilted bion' geometry is reproduced, but that the resulting D3-brane carries the correct worldvolume gauge field [77]. To this end, we make a proposal for including possible worldvolume gauge fields when mapping a noncommutative geometrical brane solution onto a corresponding commutative brane description.

In chapter 3 we study intersecting solitons in a noncommutative scalar field theory [18]. Solitons in noncommutative scalar field theory in the large noncommutativity limit were first constructed in [58]. Based on the conjecture by Sen [106], when the scalar field is interpreted as the string tachyon, the solitons can be interpreted as D-branes [69]. Noncommutative field theory provides a simple testing ground for the conjectures about the role of the open string tachyon. In chapter 3, we examine the intersections, fluctuations and deformations of codimension two solitons in field theory on noncommutative $\mathbb{R}^4$, in the limit of large noncommutativity. We find that holomorphic deformations are zero modes of flat branes, and we show that there is a zero mode localized at the intersection of two solitons.

An important recent development in string theory is the discovery of the celebrated AdS/CFT correspondence, and of gravity duals in general. Gravity duals can be used as a tool to understand nonperturbative behavior of field theories. One of the features of noncommutative field theories is that they are not local. The gravity duals of nonlocal field theories in the large N limit exhibit a novel behavior near the boundary. To explore this, in chapter 4 we present and study the duals of dipole theories [16], which are a particular class of nonlocal theories with fundamental dipole fields, simpler to study than the noncommutative field theory. The nonlocal interactions are manifest in the metric of the gravity dual. We compare the situation to that in noncommutative SYM.

Finally, in chapter 5, we study quantum mechanics on the hyper-Kähler manifold that is the blow-up of an A_1-singularity [53]. This system is relevant for the Matrix model of the five-brane in M-theory. The blow up of the singularity corresponds to a noncommutative regularization of the theory under study.

Chapter 2

The tilted noncommutative bion

This chapter is devoted to a study of the D1-D3 brane intersection in the noncommutative picture provided by the nonabelian action describing a bundle of D1-branes [77]. We will show that the noncommutative bion solution from [35] can be generalized to include a nonzero NS-NS two-form field B. The geometry extracted from our generalized solution agrees with the 'dual' picture provided by the abelian theory on a D3-brane in the presence of a nonzero B. Even better, we will be able to argue that the nonabelian calculation makes a prediction for the worldvolume gauge field on the D3-brane and we will find that this agrees with the abelian calculation as well. Although limited to BPS configurations, we can regard these considerations as a significant further test of the validity of the nonabelian Born-Infeld action proposed by Myers in [88].

2.1 Introduction

As was described in section 1.2, the low energy dynamics of a single D-brane is described by the Born-Infeld action, (1.42). As was also described there, the gauge group for a stack of N superposed D-branes is enhanced from $U(1)^N$ to $U(N)$ and the brane worldvolume supports a $U(N)$ gauge field as well as a set of scalars in the adjoint representation of $U(N)$ (one for each of the transverse coordinates). Much effort has been devoted to generalizing the Born-Infeld action so as to describe the dynamics of multiple superposed D-branes to all orders in α'. A specific proposal for a generalized action involving such fields has recently been given by Myers in [88] (see also references in this paper for other work on this problem). For a Dp-brane in static gauge, Myers' action takes the following, rather forbidding, form:

$$S_{BI} = -T_p \int d^{p+1}\sigma \; STr \;\; e^{-\phi} \;\; \times \tag{2.1}$$
$$\sqrt{-\det(P_{ab}[(G+\lambda B)_{\mu\nu} + (G+\lambda B)_{\mu i}(Q^{-1}-\delta)^{ij}(G+\lambda B)_{j\nu}] + \lambda \mathcal{F}_{ab})\det(Q^i_j)} \;,$$

where

$$Q^i_j \equiv \delta^i_j + i\lambda[\Phi^i, \Phi^k](G+\lambda B)_{kj} \;, \tag{2.2}$$

$\lambda \equiv 2\pi\alpha'$, a, b are indices in the worldvolume of the Dp-branes, i, j, k are in the transverse space and μ, ν are ten-dimensional indices. The Φ^i are $N \times N$ matrix scalar fields describing the transverse displacements of the branes and $\mathcal{F}_{ab}$ is the worldvolume gauge field (both fields are in the adjoint representation of $U(N)$). The symbol $P_{ab}[M]$ stands for a pullback of the ten-dimensional matrix M to the brane worldvolume in which the matrix coordinates Φ^i define the surface to which to pull back and all derivatives of these coordinates are made gauge covariant. Finally, STr is Tseytlin's symmetrized trace operation [115]. We refer the reader to Myers' paper [88] for a more complete definition and a discussion of useful simplifying approximations. Whether or not this action is "exact", it does seem to capture a lot of information about such structures as the commutators of the Φ's with themselves, which vanish in the $U(1)$ limit and cannot be directly inferred from the abelian Born-Infeld action.

In [35], Myers' action was applied to a stack of N D1-branes in a flat background and it was shown that the transverse coordinates of the D1-branes 'flare-out' to a flat three-dimensional space. This was interpreted as a description of a collection of D1-branes attached to an orthogonal D3-brane, one of the standard D-brane intersections. This situation has a 'dual' description in terms of a single (abelian) D3-brane carrying a point magnetic charge. Using the Born-Infeld action, one finds that a magnetic monopole of strength N produces a singularity in the D3-brane's transverse displacement, a 'spike' which can be interpreted as N D1-strings attached to the D3-brane [27, 55]. To the extent that it is possible to compare them, there is a perfect quantitative match between the two pictures. The states in question are BPS states and this match is evidence that the Myers' action (2.1) captures the dynamics of BPS states at least.

In this chapter, we extend these considerations to a more complex example of a BPS state: we generalize the solution of [35] to include a nonzero background NS-NS two-form field B, in a direction transverse to the D1-branes, but parallel to the emergent D3-brane. In the 'dual' description on the D3-brane, the B field becomes a $U(1)$ magnetic field on the D3-brane which pulls on the magnetic charge associated with the D1-brane. Simple force balance considerations suggest that the spike representing the D1-branes should be tilted away from the direction normal to the D3-brane, and this is precisely what is found in explicit solutions of the Born-Infeld equations [74, 87]. We will show that this system can be analyzed in the nonabelian D1-brane picture and that there is, once again, a perfect quantitative match between the results of the two dual calculations. One slight novelty of our approach is that we can demonstrate this agreement not only for geometrical quantities, but also for worldvolume gauge fields.

The chapter is organized as follows: In section 2.2, we review how the bion arises in the abelian theory on a D3-brane, both with and without a B-field. In section 2.3, we review the construction of the nonabelian bion, and then extend it to a nonzero B-field. In section 2.4, we take a brief detour and describe how lower dimensional D-branes can form flat, noncommutative, higher dimensional D-branes equipped with worldvolume gauge fields. In this context we show how the higher dimensional worldvolume gauge field is constructed out of the lower dimensional noncommuting coordinates. Finally, in section 2.5, we apply this recipe to the gauge field on the bion of section 2.3 and show that there is perfect agreement between the two approaches, in both the geometry and the field strength on the brane.

2.2 The Bion solution on a D3-brane

Consider the abelian Born-Infeld action for a single D3-brane in flat space. Let the D3-brane extend in 0123-directions, and let the coordinates on the brane be denoted by x^i, $i = 0, \ldots, 3$. Restricting the brane to have displacement in only one of the transverse directions, we can take the (static gauge) embedding coordinates of the brane in the ten-dimensional space to be $X^i = x^i$, $i = 0, \ldots, 3$; $X^a = 0$, $a = 4, \ldots, 8$; $X^9 = \sigma(x^i)$. Then there exists a static (BPS) solution of the Born-Infeld action corresponding to placing N units of $U(1)$ magnetic charge at the origin of coordinates on the brane [27]:

$$X^9(x^i) = \sigma(x^i) = \frac{q}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}} \ , \tag{2.3}$$

where $q = \pi\alpha' N$, and N is an integer. This magnetic bion solution to the Born–Infeld action corresponds to N superposed D1-strings attached to the D3-brane at the origin. It is "reliable" in the sense that the effect of unknown higher-order corrections in α' and g to the action can be made systematically small in the large-N limit (see [27] for details). At a fixed $X^9 = \sigma$, the cross-section of the deformed D3-brane is a 2-sphere with a radius

$$r(\sigma) = \frac{\pi\alpha' N}{\sigma} \ . \tag{2.4}$$

In the presence of a two-form field B parallel to the world volume of the brane, ($\frac{1}{2}Bdx^1 \wedge dx^2$ to be concrete), the above solution is modified as follows [87]:

$$\frac{\sigma}{\cos(\alpha)} = \frac{q}{\sqrt{(x^1)^2 + (x^2)^2 + \cos(\alpha)^2(x^3 - \tan(\alpha)\sigma)^2}} \ . \tag{2.5}$$

where $\tan(\alpha) = 2\pi\alpha' B$. Because the transverse displacement σ is not a single-valued function of the base coordinates x^i, the geometry is somewhat obscure. It is more transparent in rotated coordinates defined by

$$\begin{aligned} Y^1 &= X^1 \qquad (2.6) \\ Y^2 &= X^2 \\ Y^3 &= \cos(\alpha)X^3 - \sin(\alpha)X^9 \\ Y^4 &= \sin(\alpha)X^3 + \cos(\alpha)X^9 \ . \end{aligned}$$

Choosing $Y^{1,2,3}$ as the worldvolume coordinates, the embedding becomes

$$Y^{(1,2,3)}(y) = y^{(1,2,3)}, \quad Y^4(y) = \tan(\alpha)\ y^3 + \frac{q}{\sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}} \ . \tag{2.7}$$

It is easy to see that this describes D1-strings tilted away from the normal to the D3-brane by an angle α (when $B \to 0$, $\alpha \to 0$, and the branes become orthogonal).

Reverting to the original coordinates (2.5), we can show that the D3-brane at a fixed transverse displacement $X^9 = \sigma$ is an ellipsoid of revolution defined by the equation

$$1 = \frac{x_1^2}{r_1^2(\sigma)} + \frac{x_2^2}{r_2^2(\sigma)} + \frac{(x_3 - \sigma \tan(\alpha))^2}{r_3^2(\sigma)} \tag{2.8}$$

with major and minor axes

$$r_1(\sigma) = r_2(\sigma) = \cos(\alpha)\frac{\pi\alpha' N}{\sigma} , \qquad r_3(\sigma) = \frac{\pi\alpha' N}{\sigma} . \tag{2.9}$$

For large σ, the ellipsoid becomes small and defines a slice through the D1-brane that is attached to the D3-brane. The ellipsoid is centered at brane coordinates $(X_1, X_2, X_3) = (0, 0, \tan(\alpha)\sigma)$ and the fact that X_3 varies linearly with σ implements the tilt of the D1-brane. The tilting arises for simple reasons of force balance. The D1-brane spike behaves like a magnetic charge from the point of view of the worldvolume gauge theory; the background B field is equivalent to a uniform magnetic field on the D3-brane and exerts a force on the magnetic charge which must be balanced by a component of the D1-brane tension along the D3-brane .

The gauge field on the brane is particularly easy to obtain in the tilted coordinates (2.6). The configurations under discussion are not the most general solution of the equations of motion: they are special minimal energy solutions that satisfy the BPS condition (and preserve some supersymmetry). The BPS condition relates the total 2-form field on the brane to the divergence of the transverse displacement scalar as follows [27, 55]:

$$(2\pi\alpha')\epsilon^{ijk}(F+B)_{jk} = \pm\frac{\partial}{\partial y^i}Y^4 = \frac{\mp q}{[(y^1)^2 + (y^2)^2 + (y^3)^2]^{3/2}}\; y^i + \tan(\alpha)\delta_3^i . \tag{2.10}$$

The ambiguous sign encodes the difference between a D3- and a $\overline{\text{D3}}$-brane. From this we read off that the magnetic field $\mathcal{B}_k \equiv \epsilon^{ijk} F_{jk}$ is just

$$\mathcal{B}_i(y) = \mp \frac{N}{2[(y^1)^2 + (y^2)^2 + (y^3)^2]^{3/2}} \; y^i \; , \tag{2.11}$$

or a Coulomb field due to N charges. The end of the D1-brane(s) acts as a magnetic charge and the space-time B field provides an effective uniform magnetic field which exerts a force on the magnetic charge, hence tilting the D1-branes.

Note that for the upper sign (D3-brane), when N is positive, the D1-brane(s) run 'towards' the D3-brane, and when it is negative, they run 'away' from the D3-brane. For the lower sign ($\overline{\text{D3}}$-brane case), the end of the $\overline{\text{D1}}$-brane(s) running 'towards' the $\overline{\text{D3}}$-brane acts as a positive magnetic source and the end of the $\overline{\text{D1}}$-brane(s) running 'away' from $\overline{\text{D3}}$-brane acts as negative one. We will encounter the same four cases in our dual treatment by the nonabelian D1-branes.

2.3 Dual treatment by nonabelian D1-branes

In describing the intersection of one D3-brane with N D1-branes, one has the option of starting from the dynamics of the D3-brane and trying to derive the D1-branes (this was the approach of the previous section), or of starting from the nonabelian dynamics of multiple D1-branes and trying to derive the D3-brane. The latter approach has been applied in [35] to the case in which there is no background B field. In this section we will review that work and show how to generalize it to the case where $B \neq 0$ and the bion is tilted.

We begin by reviewing the results from [35] on the $B = 0$ case, while introducing some notation which will be useful later. We consider the nonabelian Born–Infeld action of equation (2.1) specialized to the case of N coincident D1-branes, flat background spacetime ($G_{\mu\nu} = \eta_{\mu\nu}$), vanishing B field, vanishing worldvolume gauge field and constant dilaton. The action then depends only on the $N \times N$ matrix transverse scalar fields Φ^i's. In general, $i = 1, \ldots, 8$, but since we are interested in studying the D1/D3-brane intersection, we will allow only three transverse coordinate fields to be active ($i = 1, 2, 3$). The explicit reduction of the static gauge action ($X^0 = \tau$ and $X^9 = \sigma$) is then

$$S_{BI} = -T_1 \int d\sigma d\tau STr \sqrt{-\det(\eta_{ab} + \lambda^2 \partial_a \Phi^i Q^{-1}_{ij} \partial_b \Phi^j) det(Q^{ij})} \ , \tag{2.12}$$

where

$$Q^{ij} = \delta^{ij} + i\lambda[\Phi^i, \Phi^j] \ . \tag{2.13}$$

Since the dilaton is constant, we incorporate it in the tension T_1 as a factor of g^{-1}.

We look for static solutions (*i.e.*, $\Phi = \Phi(\sigma)$ only). Since we have no hope of finding a general static solution of these nonlinear matrix equations, we make some simplifying assumptions which have a chance of being valid on the restricted class of BPS solutions. The action (2.12) depends only on the two matrix structures $\partial_a \Phi^i$ and $W_i \equiv \frac{1}{2} i \epsilon_{ijk} [\Phi^j, \Phi^k]$ and, because of the nature of the STr instruction, they may be treated as commuting quantities until the final step of doing the gauge trace to evaluate the action. This allows us to evaluate the determinants in the definition of the action (2.12) and to convert the energy functional to the following form:

$$U_{B=0} = \int d\sigma STr \sqrt{1 + \lambda^2 (\partial \Phi^i)^2 + \lambda^2 (W_i)^2 + \lambda^4 (\partial \Phi^i W_i)^2} \ . \tag{2.14}$$

Continuing to treat Φ^i and W^i as commuting objects, we see that this energy functional can be written as a sum of two squares

$$U_{B=0} = \int d\sigma STr \sqrt{(1 \pm \lambda^2 \partial \Phi^i W_i)^2 + \lambda^2 (\partial \Phi^i \mp W_i)^2} \ , \tag{2.15}$$

and is minimized by a displacement field satisfying the first-order BPS-like equation $\partial\Phi^i = \pm W_i$. This equation, written more explicitly as

$$\partial\Phi_i = \pm\frac{1}{2}i\epsilon_{ijk}[\Phi^j,\Phi^k] \ , \tag{2.16}$$

is known as the Nahm equation [40]. The $\pm$ ultimately corresponds to the choice between a bundle of D1- or $\overline{\text{D1}}$-branes. The Nahm equation is a very plausible candidate for the exact equation to be satisfied by a BPS solution of this system and the fact that the Myers action (2.12) implies it in the BPS limit is very satisfactory.

The Nahm equation has the trivial solution $\Phi = 0$ which corresponds to an infinitely long bundle of coincident D1-branes. In [35], a much more interesting solution was found by starting with the following ansatz:

$$\Phi^i = \hat{R}(\sigma)\alpha^i, \qquad\qquad (\alpha^1,\alpha^2,\alpha^3) \equiv \mathbf{X} \ , \tag{2.17}$$

where α^i form an $N \times N$ representation of the generators of an $SU(2)$ subgroup of $U(N)$, $[\alpha^i,\alpha^j] = 2i\epsilon_{ijk}\alpha^k$. With this ansatz, both $\partial\Phi^i$ and W^i are proportional to the generator matrix α^i. When the ansatz is substituted into the BPS condition (2.16), we obtain a simple equation for $\hat{R}$,

$$\hat{R}' = \mp 2\hat{R}^2 \ , \tag{2.18}$$

which is solved by

$$\hat{R} = \pm\frac{1}{2\sigma} \ . \tag{2.19}$$

Substituting the ansatz (2.17) into (2.14) leads to the following effective action for

$\hat{R}(\sigma)$:

$$U_{B=0}[\hat{R}(\sigma)] = \int d\sigma STr\sqrt{(1+\lambda^2(\hat{R}')^2\mathbf{X}^2)(1+4\lambda^2(\hat{R})^4\mathbf{X}^2)} \ . \tag{2.20}$$

It can be shown that (2.19) satisfies the equations of motion following from the action (2.20).

This solution maps very nicely onto the bion solution of the previous section. At a fixed point $|\sigma|$ on the D1-brane stack, the geometry given by (2.17) is that of a sphere with the physical radius $R^2 = \frac{\lambda^2}{N}Tr(\Phi^i)^2$ (the only sensible way to pass from the matrix transverse displacement field $\lambda\Phi^i$ to a pure number describing the geometry). For the ansatz under consideration, this gives

$$R(\sigma)^2 = \frac{\lambda^2}{N}Tr(\Phi^i)^2 = \lambda^2\hat{R}(\sigma)^2 C \ , \tag{2.21}$$

where C is the quadratic Casimir, equal to N^2-1 for an irreducible representation of $SU(2)$. This gives

$$R(\sigma) = \frac{\lambda\sqrt{N^2-1}}{(2|\sigma|)} \cong \frac{\pi\alpha' N}{|\sigma|} \tag{2.22}$$

for large N, in agreement with equation (2.4). This completes our synopsis of the arguments given in [35] for the agreement between the commutative and noncommutative approaches to the D1/D3-brane intersection.

A simple argument can be made at this point to strengthen the meaning of equation (2.22). At a fixed σ, the Fourier transform of the density of the $D1$-strings is given by [112, 113]

$$\widetilde{\rho}(k) = Tr\left(e^{i\lambda k_i\Phi^i}\right) \ . \tag{2.23}$$

This is simply the operator to which the 09-component of the RR 2-form $C^{(2)}$, C_{09}, couples. For the solution (2.17), this is evaluated to give

$$\widetilde{\rho}(k) = \frac{\sin(\lambda N \hat{R}|k|)}{\sin(\lambda \hat{R}|k|)} \ , \tag{2.24}$$

which for $N \gg 1$, and k such that $(\lambda N\hat{R})^{-1} < |k| \ll (\lambda\hat{R})^{-1}$ (i.e., for momentum large enough to resolve the size of the sphere, but not large enough to resolve the individual brane constituents) gives

$$\widetilde{\rho}(k) \cong \frac{\sin(\lambda N \hat{R}|k|)}{\lambda \hat{R}|k|} \ . \tag{2.25}$$

This is precisely the Fourier transform of the density distribution representing a thin shell,

$$\begin{aligned} \widetilde{\rho}(k) &= \int d^3x e^{i\vec{k}\cdot\vec{x}} \rho(x) \\ \text{for} \quad \rho(x) &= \frac{N}{4\pi R^2}\delta(|x|-R) \quad \text{with} \quad R = \lambda N \hat{R} = \frac{\pi\alpha' N}{|\sigma|} \ , \end{aligned} \tag{2.26}$$

in agreement with equation (2.22).

To distinguish the various cases involving branes and antibranes, note that σ can either run from $-\infty$ to 0, in which case the D1($\overline{\text{D1}}$)-branes run 'towards' the D3($\overline{\text{D3}}$)-brane plane, or from 0 to ∞, in which case the D1($\overline{\text{D1}}$)-branes run 'away' the D3($\overline{\text{D3}}$)-brane plane. We have thus four cases:

A : Stack of D1-branes expanding to a D3-brane , $\partial\Phi^i = +W_i$ and $\sigma \in (-\infty, 0)$

B : Stack of D1-branes expanding to a D3-brane , $\partial\Phi^i = +W_i$ and $\sigma \in (0, \infty)$

C : Stack of $\overline{\text{D1}}$-branes expanding to a $\overline{\text{D3}}$-brane , $\partial\Phi^i = -W_i$ and $\sigma \in (-\infty, 0)$

D : Stack of $\overline{\text{D1}}$-branes expanding to a $\overline{\text{D3}}$-brane , $\partial\Phi^i = -W_i$ and $\sigma \in (0, \infty)$

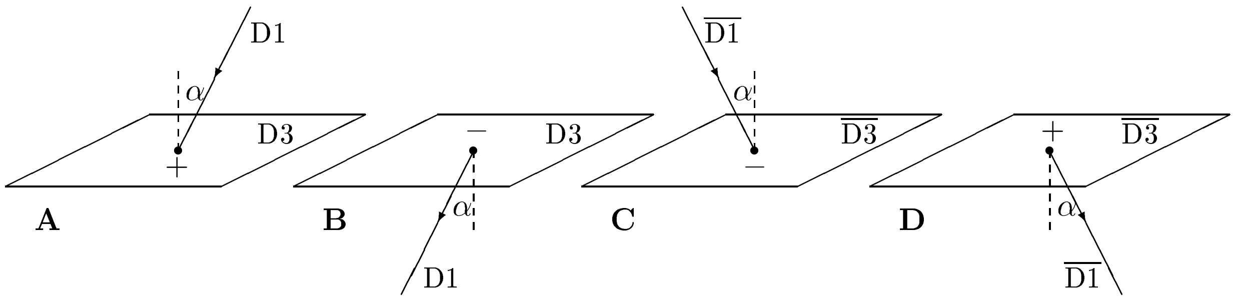


Figure 2.1: The four cases **A**-**D** discussed in the text.

Cases **A** and **D** correspond to the D1($\overline{\text{D1}}$)-branes running 'towards' ('away from') the D3($\overline{\text{D3}}$)-brane plane, and thus should represent a positive magnetic charge, while Cases **B** and **C** should represent a negative magnetic charge. We will see shortly that this is the case and that the four cases match the four cases found in the abelian treatment of the same problem starting from the D3-brane (see the end of section 2.2).

Our next step is to turn on the background B field in order to study the tilted bion from the noncommutative D1-brane point of view. The only difference from the previous case is that we turn on the component B_{12} = const. of the background B field (remember that the D1-brane worldvolume spans the (X^0, X^9) plane and that only the $i = 1, 2, 3$ components of the matrix transverse displacement field are allowed to be nonzero). It is still the case that the action is a functional only of the fields $\partial\Phi^i$ and $\epsilon_{ijk}[\Phi^j, \Phi^k]$ and the same reasoning as before leads us to treat these quantities as commuting objects inside the STr instruction. In this way, the action (2.1) can be reduced to something much more explicit. To get the most transparent results, it helps to define rescaled fields

$$\varphi_1 = \sqrt{(1+\lambda^2 B^2)}\Phi_1 \ , \qquad \varphi_2 = \sqrt{(1+\lambda^2 B^2)}\Phi_2 \ , \qquad \varphi_3 = \Phi_3 \ , \tag{2.27}$$

and to redefine the commutator W^i as

$$W_i \equiv \frac{1}{2} i\epsilon_{ijk}[\varphi^j, \varphi^k] - \delta_i^3 B \ . \tag{2.28}$$

After some rather tedious algebra (made much easier by MAPLE) to evaluate the determinants in the definition of the action, we get a result for the energy functional that is almost identical to (2.14):

$$\begin{aligned} U_{B\neq 0} &= \frac{1}{\sqrt{1+\lambda^2 B^2}} \int d\sigma STr \sqrt{1+\lambda^2(\partial\varphi^i)^2 + \lambda^2(W_i)^2 + \lambda^4(\partial\varphi^i W_i)^2} \\ &= \frac{1}{\sqrt{1+\lambda^2 B^2}} \int d\sigma STr \sqrt{(1 \pm \lambda^2 \partial\varphi^i W_i)^2 + \lambda^2(\partial\varphi^i \mp W_i)^2} \ . \end{aligned} \tag{2.29}$$

The action is still 'linearized' by taking $\partial\varphi^i = \pm W_i$, which means that the BPS condition in the presence of a background B field is

$$\partial\varphi_i = \pm i(\frac{1}{2}\epsilon_{ijk}[\varphi^j, \varphi^k] + \delta_i^3 iB) \ . \tag{2.30}$$

This is precisely the generalization of the Nahm equation that has been derived in the context of studies of magnetic monopoles in noncommutative field theory [10]. It is a plausible candidate for the exact BPS condition for the nonabelian D1-brane system and we will show that it gives a detailed account of the physics of the tilted bion. The fact that the generalized Nahm equation is implied by the Myers action is further evidence for the essential correctness of the latter.

In order to solve the modified equations of motion, we have to slightly modify the ansatz (2.17), expressing the fields φ in terms of generators of an N-dimensional representation of $SU(2)$ and a scalar function $\hat{R}(\sigma)$:

$$\varphi^i = \hat{R}(\sigma)\alpha^i - \delta_i^3 \frac{B}{2\hat{R}(\sigma)} \ . \tag{2.31}$$

When this modified ansatz is substituted into the BPS equation (2.30), we obtain the same equation for $\hat{R}$ as before, namely $\hat{R}' = \mp 2\hat{R}^2$: solution (2.19) still holds. If we collect the generators into a modified triplet $\mathbf{X} \equiv (\alpha^1, \alpha^2, \alpha^3 + \frac{B}{2\hat{R}^2})$ and use (2.31), the action (2.29) can be expressed as an effective action for $\hat{R}(\sigma)$:

$$U_{B\neq 0}[\hat{R}(\sigma)] = \frac{1}{\sqrt{1+\lambda^2 B^2}} \int d\sigma STr\sqrt{(1+\lambda^2(\hat{R}')^2\mathbf{X}^2)(1+4\lambda^2(\hat{R})^4\mathbf{X}^2)} \ . \tag{2.32}$$

This looks the same as (2.20) but is not quite because $\mathbf{X}$ now depends on $\hat{R}$. Nevertheless, the same radial function (2.19) continues to be a solution, further testing the compatibility of the action with the BPS condition. Notice that this does not conclusively prove that the ansatz (2.31) with (2.19) is a solution to the full equations of motion implied by (2.29).

It is easily seen that this solution corresponds to the tilted bion solution discussed in the previous section. Equation (2.27) matches the ratios of axes given in (2.9). The over-all size of the spheroid agrees with (2.9) by an argument identical to that given for $B = 0$. The shift of its center is given by

$$\Delta^i(\sigma) = \frac{1}{N} tr(\lambda\Phi^i) = \Delta\delta_3^i \tag{2.33}$$

(by virtue of the fact that $tr(\alpha^i) = 0$), where

$$\Delta = -\frac{\lambda B}{2\hat{R}} = \mp\lambda B\sigma = \begin{cases} \tan(\alpha)|\sigma| & \text{for cases } \mathbf{A} \text{ and } \mathbf{D}. \\ -\tan(\alpha)|\sigma| & \text{for cases } \mathbf{B} \text{ and } \mathbf{C}. \end{cases} \tag{2.34}$$

Thus, in cases **A** and **D**, the bion tilts in agreement with section 2.2. In the other two cases, it tilts in the opposite direction. The interpretation is that D1-branes coming 'towards' the $D3$-brane correspond to a positively charged magnetic monopole, while D1-branes coming 'away from' the $D3$-brane correspond to a negatively charged one. Similarly, $\overline{\text{D1}}$-strings coming 'away from' the $D3$-brane correspond to a positively charged magnetic monopole, while D1-branes coming 'towards' the $D3$-brane correspond to a negatively charged one. The geometry of the tilted bion inferred from the nonabelian dynamics of D1-branes perfectly matches the results of the abelian D3-brane calculation summarized in the previous section.

2.4 Flat D$(p+r)$-brane from Dp-branes

It is known that within Yang-Mills theory, lower dimensional branes can expand to form higher dimensional noncommutative branes (see, for example, [103] and references therein). In this section, we show how this construction can be extended to the nonlinear case of the nonabelian BI action. The point of this exercise (which looks like a detour from the line of argument of the rest of the chapter) is to infer a specific recipe for evaluating the worldvolume gauge fields in the noncommutative description of a D-brane. In the next section we will apply the spirit of this recipe to the curved branes which are our primary interest.

We take the spacetime metric to be the flat Minkowski metric ($g_{\mu\nu} = \eta_{\mu\nu}$), the dilaton to be constant and the worldvolume gauge field to be zero. We take the background two-form field B to have nonzero (constant) components only in directions transverse to the brane, $i, j, k = p+1, \ldots, 9$. The world-volume of the branes is parametrized by $X^a = \sigma^a, a = 0, \ldots, p$ (*i.e.* we are using the static gauge) and the transverse fluctuations are $X^i = \lambda \Phi^i$, where Φ^i are $N \times N$ matrices in the adjoint of the gauge group.

We will look for solutions where the transverse scalars Φ are not functions of the brane coordinates σ^a. In this case, the action (2.1) for the nonabelian dynamics of N Dp-branes reduces to

$$S = -gT_p \int d^p x^a STr\left(\sqrt{det(Q^i_j)}\right) \ . \tag{2.35}$$

where the explicit form of Q is displayed in (2.2). It is easy to to show that matrices Φ^i, satisfying

$$[\Phi^i, \Phi^j] = \frac{i}{\lambda^2}\theta^{ij} I_{N\times N} \ , \tag{2.36}$$

where θ^{ij} is an arbitrary $(9-p)\times(9-p)$ antisymmetric matrix of c-numbers, solve the equations of motion. Substituting this solution into the general definition of Q, (2.2), gives

$$Q^i_j = \delta^i_j - \lambda^{-1}\theta^{il}(g+\lambda B)_{lj} \ . \tag{2.37}$$

θ can have any even rank r up to $9-p$. With no loss of generality, we can block diagonalize θ so that $\theta^{\mu\nu} \neq 0$ for $\mu, \nu = p+1, \ldots, p+r$. The remaining directions will be denoted by $m, n = p+r+1, \ldots, 9$ and of course $\theta^{in} = 0$ for i, n in their appropriate ranges. From now on, $\theta = \theta^{\mu\nu}$ will denote an invertible, $r \times r$ matrix with inverse $\theta_{\mu\nu}$. Also, let us restrict our attention to background two-form fields B only in directions $p+1, \ldots, p+r$, since the other components can simply be gauged away.

The above can be summarized by saying that solution (2.36) divides the Φ^i's into $r/2$ pairs satisfying canonical commutation relations and $9-p-r$ other commuting coordinates (which we can drop from further consideration). The slight hitch is that canonical commutation relations can only be realized on infinite-dimensional function spaces, and not on finite-dimensional matrices. The solution only makes sense if we take $N \to \infty$ and reinterpret all matrix operations (multiplication, trace, etc.) in the action as the corresponding Hilbert space operations. Fortunately, the technology for doing this sort of thing has been worked out in the study of noncommutative field theory over the past couple of years and has been briefly described in section 1.1. Indeed, the physics of small fluctuations about (2.36) is best described by a noncommutative field theory on the r-dimensional base space spanned by the noncommuting Φ^i. It is in this sense that we will interpret the existence of r noncommuting transverse displacement fields on the Dp-brane as creating an effective D(p+r)-brane. We will first show that the energetics of (2.36) are indistinguishable from that of an abelian D(p+r)-brane with a certain worldvolume gauge field (which is the quantity that is the focus of our interest).

To establish the desired result, we make use of an equivalence described in the introduction in section 1.1 between actions built on ordinary integrals of functions of ordinary coordinates, but with a noncommutative definition of multiplication of functions (the $*$-product or Moyal product) and actions where functions become operators on a harmonic oscillator Hilbert space and the action is computed as the trace of an operator on that Hilbert space (integral over space becomes Hilbert space trace) [34, 58]. In our interpretation of (2.36), the STr operation in the action (2.35) is to be thought of as a trace over operators on a Hilbert space. As in equation (1.27) , the trace can be recast as an integral over the associated noncommuting coordinates, but we need the exact relative normalization. The identification we need has been worked out in the noncommutative field theory literature [103]:

$$d^r x^\mu \longleftrightarrow (2\pi)^{\frac{r}{2}} \mathrm{Pf}(\theta) STr \ , \tag{2.38}$$

where Pf denotes the Pfaffian: $\mathrm{Pf}(\theta)^2 = det(\theta)$. Putting the various pieces of the puzzle together, we can express the action for our solution in the form of an equivalent integral of an energy density over a D(r+p)-brane worldsheet:

$$
\begin{aligned}
S(\theta) &= -gT_p \int d^p x^a d^r x^\mu \frac{1}{\sqrt{2\pi det(\theta)}} \left(\sqrt{det(\delta^i_j - \lambda^{-1}\theta^{il}(g+\lambda B)_{lj})} \right) \\
&= -gT_p (2\pi\lambda)^{-\frac{r}{2}} \int d^p x^a d^r x^\mu \left(\sqrt{det(g+\lambda B - \lambda\theta^{-1})} \right) \ . \qquad (2.39)
\end{aligned}
$$

Since $T_{p+r} = T_p(2\pi\lambda)^{-\frac{r}{2}}$, the object we have constructed has the right energy density to be a D(p+r)-brane with a world-volume F-flux equal to $-\theta^{-1}$. Noncommutative coordinates for a lower dimensional brane have, in a simple context, been converted to a higher dimensional brane carrying a worldvolume gauge field. For future reference, the interesting thing is the way the gauge field arises via the commutator of the lower-dimensional matrix coordinates (2.36).

To give this idea a more demanding test, we will now check whether we can reproduce the correct Chern-Simon couplings. We will need the nonabelian Chern-Simons action proposed by Myers in [88]

$$
S_{CS} = \mu_p \int STr\left(P\left[e^{i\lambda i_\Phi i_\Phi} (\Sigma C^{(n)} e^{\lambda B}) \right] e^{\lambda F} \right) \ , \qquad (2.40)
$$

where i_Φ is an interior product by Φ^i, for example $i_\Phi i_\Phi A^{(2)} = \frac{1}{2} A^{(2)}_{ij} [\Phi^i, \Phi^j]$. (We refer the reader to Myers' paper for the precise definition of the symbol $\lambda i_\Phi i_\Phi$ and other notation). For concreteness, specialize to p=1 (nonabelian D-strings extending in the 01-directions), with θ and B nonzero only in the 23-directions. Examine the coupling to $C^{(2)}$, specifically, to the C_{01} component. Expand the action (2.40) and pick off the coupling of interest to obtain

$$
S = \mu_1 \int dx^0 dx^1 STr C_{01} \left((1 + (i\lambda i_\Phi i_\Phi)\lambda B \right) \ . \qquad (2.41)
$$

Using the solution (2.36) for Φ, we have $(i\lambda i_\Phi i_\Phi)\lambda B = i\lambda^2[\Phi^1, \Phi^2]B_{12} = -\theta B$. Passing from STr to $\int$ according to (2.38) we obtain an expression for this interaction in terms of an integral over the D$(p+r)$-brane worldvolume:

$$S = \mu_3 \int dx^0 dx^1 dx^2 dx^3 C_{01} \left(B - \theta^{-1}\right). \tag{2.42}$$

This is precisely the right coupling for a D3-brane with a world-volume field $F_{23} = -\theta^{-1}$. The world-volume field F suggested here corresponds precisely to the field which one would expect to get from the D-strings dissolved in a D3-brane with density θ^{-1} per area normal to the D-strings. Solution (2.36) corresponds to exactly this density of D-strings.

The lesson we learn from this computation is that when Dp-branes expand to form a D$(p+r)$-brane, the world volume gauge field F on the D$(p+r)$-brane can be computed from the inverse of the density of Dp-branes, which in turn can be obtained from the commutators of the transverse coordinates.

2.5 The Worldvolume Gauge Field on the Dual Bion

In this section, we will take the prescription given in section 2.4 for identifying the worldvolume gauge field implicit in a set of noncommuting coordinates and adapt it to the bion problem under discussion.

We begin with the simple example of N D1-branes (N large) with $B = 0$ (the set-up of section 2.3). Choose the solution (2.17) based on the $N \times N$ representation of $SU(2)$ and further specialize to the representation where α^3 is diagonal: $\alpha^3 = \mathrm{diag}(N-1, N-3, N-5, \ldots, -N+3, -N+1)$. The sphere described by equation

(2.17) at a fixed σ goes through a point $(X^1, X^2, X^3) = (0, 0, R = \pi\alpha' N/|\sigma|)$. A small patch of the sphere near this point is described by the corner $k \times k$ blocks of the full $SU(2)$ representation matrixes α^i, where $k \ll N$. Explicitly, replace α^3 with $\mathrm{diag}(N-1, N-3, \ldots, N-2k+1)$, which for $k \ll N$ is approximately just $N\ I_{k\times k}$. The small patch of the sphere is now described by the same commutator as the noncommutative plane (section 2.4),

$$[\Phi^1, \Phi^2] = i(2\hat{R})\Phi^3 \to \pm i\frac{N}{2\sigma^2}\ . \tag{2.43}$$

The '+'-sign corresponds to cases **B** and **C**, and the '−'-sign corresponds to cases **A** and **D** in section 2.3. Following section 2.4, we now write

$$\theta^{12} = -i\lambda^2[\Phi^1, \Phi^2] = \frac{\lambda^2 N}{2\sigma^2}\ , \tag{2.44}$$

so that the identification $F = -\theta^{-1}$ gives

$$F_{12} = \pm\frac{2\sigma^2}{\lambda^2 N} = \pm\frac{N}{2R^2}\ . \tag{2.45}$$

Referring back to (2.11) in section 2.2, we see that this is indeed the correct value of the worldvolume gauge field. Again, in cases **A** and **D**, we obtain the '−'-sign while in cases **B** and **C**, we obtain the opposite sign and monopole charge. This is all in agreement with expectations from (2.11). The essence of this computation is that the commutator $[\Phi^i, \Phi^j]$ defines a two-form field in the worldvolume of the D3-brane, whose inverse is the worldvolume gauge field F.

Computing the worldvolume gauge field in case of $B \neq 0$ is very similar, except the geometry is more complicated. To avoid any formulas with multiple $\pm$ signs, we will specialize to case **A** above, choosing $\sigma < 0$ and $\hat{R} = (2\sigma)^{-1}$ (the other three cases are similar). We want to evaluate the gauge field on the tilted D3-brane implied by the nonabelian solution (2.31) and compare it to the result of a direct calculation given in (2.11). To do this, it is best to convert (2.31) to the coordinates given in (2.6). Defining rotated variables

$$\begin{aligned} \Psi^1 &= \Phi^1 \,, \\ \Psi^2 &= \Phi^2 \,, \\ \Psi^3 &= \cos(\alpha)\Phi^3 - \sin(\alpha)\sigma/\lambda \,, \\ \Psi^4 &= \sin(\alpha)\Phi^3 + \cos(\alpha)\sigma/\lambda \,, \end{aligned} \tag{2.46}$$

and then inserting (2.31) gives

$$\begin{aligned} \Psi^1 &= \cos(\alpha)\frac{1}{2\sigma}\ \alpha^1 \,, \\ \Psi^2 &= \cos(\alpha)\frac{1}{2\sigma}\ \alpha^2 \,, \\ \Psi^3 &= \cos(\alpha)\frac{1}{2\sigma}\ \alpha^3 \,, \\ \Psi^4 &= \sin(\alpha)\frac{1}{2\sigma}\ \alpha^3 + \frac{1}{\cos(\alpha)}\ \frac{\sigma}{\lambda} \,. \end{aligned} \tag{2.47}$$

To check that this makes sense, take N large and pass to the classical limit, by setting

$\alpha^i \to Nn^i$ where $(n^1)^2 + (n^2)^2 + (n^3)^2 = 1$. The Ψ's become classical coordinates

$$
\begin{aligned}
\lambda\Psi^1 &\to Y^1 = \cos(\alpha)\frac{N\pi\alpha'}{\sigma}\, n^1 \ , \\
\lambda\Psi^2 &\to Y^2 = \cos(\alpha)\frac{N\pi\alpha'}{\sigma}\, n^2 \ , \\
\lambda\Psi^3 &\to Y^3 = \cos(\alpha)\frac{N\pi\alpha'}{\sigma}\, n^3 \ , \\
\lambda\Psi^4 &\to Y^4 = \sin(\alpha)\frac{N\pi\alpha'}{\sigma}\, n^3 + \frac{1}{\cos(\alpha)}\, \sigma \\
&= \tan(\alpha)Y_3 + \frac{N\pi\alpha'}{\sqrt{Y_1^2 + Y_2^2 + Y_3^3}} \ ,
\end{aligned}
\tag{2.48}
$$

in perfect correspondence with equation (2.7).

In section 2.4 we showed that the worldvolume gauge field is computed from the commutators of the transverse scalars. Using (2.47) to compute the commutators and comparing with (2.36), we obtain the following noncommutativity tensor Θ:

$$
\begin{aligned}
-i\lambda^2[\Psi^i, \Psi^j] = 2\epsilon^{ijk}\frac{\lambda^2\cos(\alpha)}{2\sigma}\Psi^k &\to \frac{\lambda\cos(\alpha)}{\sigma}\epsilon^{ijk}Y^k \equiv \Theta^{ij} \ , \\
-i\lambda^2[\Psi^1, \Psi^4] = -2\frac{\lambda^2\sin(\alpha)}{2\sigma}\Psi^2 &\to -\frac{\lambda\sin(\alpha)}{\sigma}Y^2 \equiv \Theta^{14} \ , \\
-i\lambda^2[\Psi^2, \Psi^4] = 2\frac{\lambda^2\sin(\alpha)}{2\sigma}\Psi^1 &\to \frac{\lambda\sin(\alpha)}{\sigma}Y^1 \equiv \Theta^{24} \ , \\
-i\lambda^2[\Psi^3, \Psi^4] = 0 &\to 0 \equiv \Theta^{34} \ ,
\end{aligned}
\tag{2.49}
$$

where $i, j, k = 1, \ldots, 3$. Finally, we need to pull the two-tensor Θ back to the worldvolume of the D3-brane, to the worldvolume coordinates of (2.7). With a little algebra, it can be checked that $\Theta^{\mu\nu}$ $(\mu, \nu = 1, \ldots, 4)$ satisfies

$$
\Theta^{\mu\nu} = \frac{\partial Y^\mu}{\partial y^i}\frac{\partial Y^\nu}{\partial y^j}\theta^{ij} \ , \tag{2.50}
$$

where

$$\theta^{ij} = \frac{\lambda \cos(\alpha)}{\sigma}\, \epsilon^{ijk} y^k = -\frac{2\sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}}{N}\, \epsilon^{ijk} y^k \tag{2.51}$$

(the minus sign is a consequence of our having chosen case **A**, $\sigma < 0$). According to section 2.4, the worldvolume gauge field is the negative inverse of the noncommutativity tensor θ^{ij}. However, the tensor of (2.51) is not invertible: it acts in three dimensions and has one zero eigenvalue. Following section 2.4, the inversion of θ^{ij} is to be carried out on the subspace orthogonal to the subspace of zero eigenvalues. With this understanding, we obtain the following result for the gauge field on the D3-brane

$$F_{ij} = (-\theta^{-1})_{ij} = -\frac{N}{2[(y^1)^2 + (y^2)^2 + (y^3)^2]^{3/2}}\, \epsilon^{ijk} y^k \ , \tag{2.52}$$

in perfect agreement with the abelian D3-brane result, equation (2.11). This is what we wanted to show.

Chapter 3

Noncommutative solitons

In this chapter we will study the fluctuations of a two-dimensional soliton (2-brane) in scalar noncommutative field theory in four dimensions, in the limit of large noncommutativity. When the scalar field is taken to be the string tachyon, these solitons find an interpretation in string theory as D-branes. On the geometrical level, a plane in four dimensions can be deformed into a curved minimal area surface, and we find that holomorphic deformations are the zero modes of flat branes when described as noncommutative solitons.

We also begin the study of intersecting 2-branes in this theory. Geometrically, two intersecting planes can be deformed into a single smooth minimal area surface. In string theory, the latter phenomenon is related to a zero-mode which appears at the intersection of two D-branes [105]. We show that there is a zero mode localized at the intersection of two solitons.

Following [58], we will work in the large noncommutativity limit but include the kinetic energy to first order.

The chapter is organized as follows. In section 3.1 we review previous work leading to the construction of D-branes as noncommutative solitons. In section 3.2 we study the deformation modes of a single 2-brane in 4D scalar NCFT. In section 3.3 we review the geometry of the deformation of intersecting planes. We will show that half of the deformation modes correspond to the deformations of the flat 2-brane into a holomorphic curve embedded in $\mathbb{R}^4$. The other half correspond to anti-holomorphic fluctuations. In section 3.4 we describe the solution corresponding to two intersecting branes and study the zero-modes that correspond to their deformations. In section 3.5 some extensions to the case of multiple branes and more dimensions are discussed.

3.1 Noncommutative solitons, tachyon condensation and D-branes

In this section, we review some previous work on noncommutative solitons in scalar field theories in the limit of large noncommutativity [58], and on tachyon condensation [106]. These two ideas will then be put together, interpreting the noncommutative scalar field as the string tachyon, and the solitons as D-branes [66, 70, 38, 69, 68, 71].

3.1.1 Noncommutative solitons

First, let us review the construction of [58] for a single codimension-2 brane in the theory with action:

$$\int [(\partial_\mu \Phi)^2 + V(\Phi)] \ . \tag{3.1}$$

Here:

$$V(\lambda) = \sum_{n=2}^{\infty} a_n \lambda^n, \qquad V(\Phi) = a_2 \Phi \star \Phi + a_3 \Phi \star \Phi \star \Phi + \cdots \ . \tag{3.2}$$

We take spacetime to be commutative and define the $\star$-product as in equation (1.23):

$$\Phi \star \Psi \equiv \Phi e^{\frac{i\theta}{2} \frac{\overleftarrow{\partial}}{\partial x_1} \frac{\overrightarrow{\partial}}{\partial y_1} - \frac{i\theta}{2} \frac{\overleftarrow{\partial}}{\partial y_1} \frac{\overrightarrow{\partial}}{\partial x_1}} \Psi \ , \tag{3.3}$$

so that:

$$x_1 \star y_1 - y_1 \star x_1 = i\theta \ . \tag{3.4}$$

Taking the limit $\theta \to \infty$ and rescaling the coordinates, the kinetic term is of order $1/\theta$ and can be neglected. For now, the x_2, y_2 coordinates are still commutative.

We set $z_1 = x_1 + iy_1$ and define a Hilbert space $\mathcal{H}_1$ with the harmonic oscillator basis, $|n\rangle$ for $n = 0, 1, \ldots$, such that $\hat{a}_1^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $\hat{a}_1|n\rangle = \sqrt{n}|n-1\rangle$. If Φ is a function of x_1 and y_1, the Weyl formula (1.21) transforms it into an operator on this Hilbert space:

$$\hat{\Phi} \equiv \frac{1}{2\pi} \int d^2\zeta \, \Phi(z_1, \overline{z}_1) e^{i\zeta\overline{z}_1 - i\overline{\zeta}z_1} \ . \tag{3.5}$$

Then $z_1 \to \sqrt{2\theta}\hat{a}_1$ and $\overline{z}_1 \to \sqrt{2\theta}\hat{a}_1^\dagger$. From now on, $\Phi, \Psi, \ldots$ will denote ordinary functions and $\hat{\Phi}, \hat{\Psi}, \ldots$ will denote the corresponding operators.

Let us assume that $V(\Phi)$ has a minimum at $\lambda \neq 0$. One can then construct a soliton by setting:

$$\hat{\Phi} = \lambda \hat{P}, \qquad \hat{P}^2 = \hat{P} \ . \tag{3.6}$$

The operator $\hat{\Phi}$ satisfies $V(\hat{\Phi}) = V(\lambda)\hat{P}$ and hence $V'(\hat{\Phi}) = 0$. The corresponding (Weyl transformed) solution, Φ, is constant in the z_2 direction. For any unitary operator, $\hat{U}$, V'$(\hat{U}^\dagger\hat{\Phi}\hat{U})$ is also zero.

If we include the kinetic term, only the operators of the form

$$\hat{P} = |\alpha\rangle\langle\alpha|, \qquad |\alpha\rangle \equiv e^{\alpha\hat{a}_1^\dagger - \overline{\alpha}\hat{a}_1}|0\rangle \ , \tag{3.7}$$

corresponding to projections onto a coherent state of the harmonic oscillator, remain as good solitons. To see this we can write the kinetic energy as

$$\begin{aligned} K &= -\frac{1}{2\theta^2}\mathrm{tr}\{[\hat{x}_1,\hat{\Phi}]^2+[\hat{y}_1,\hat{\Phi}]^2\} \\ \hat{x}_1 &\equiv \sqrt{\frac{\theta}{2}}(\hat{a}_1+\hat{a}_1^\dagger), \quad \hat{y}_1\equiv -i\sqrt{\frac{\theta}{2}}(\hat{a}_1-\hat{a}_1^\dagger)\ . \end{aligned} \tag{3.8}$$

For $\hat{P}=|\phi\rangle\langle\phi|$, we find

$$\frac{\theta^2}{\lambda^2}K=\Delta x_1^2+\Delta y_1^2\ , \tag{3.9}$$

where

$$\Delta x_1^2=\langle\phi|\hat{x}_1^2|\phi\rangle-\langle\phi|\hat{x}_1|\phi\rangle^2\ ,\qquad \Delta y_1^2=\langle\phi|\hat{y}_1^2|\phi\rangle-\langle\phi|\hat{y}_1|\phi\rangle^2 \tag{3.10}$$

are the uncertainties in $\hat{x}_1$ and $\hat{y}_1$. Now we can see that the coherent states, $|\alpha\rangle$, minimize the kinetic energy. This is because:

$$\Delta x_1^2+\Delta y_1^2\geq 2\Delta x_1\Delta y_1\geq 1\ , \tag{3.11}$$

and the equalities hold only for a coherent state. Thus, in the space of all possible unitary transformations, $\hat{U}$, acting on $\hat{\Phi}$, the kinetic energy has flat directions corresponding to translating the brane rigidly in the z_1 direction.

The results of [58], some of which were summarized here, have been extended to gauge theories in [61, 97, 60, 11, 2, 67, 62, 12, 91].

Now, let us add two extra noncommutative directions:

$$x_1\star y_1-y_1\star x_1=x_2\star y_2-y_2\star x_2=i\theta\ . \tag{3.12}$$

As with z_1, z_2 corresponds to an operator on a Hilbert space $\mathcal{H}_2$. Φ, as a function of x_1, y_1, x_2 and y_2, corresponds to an operator on the Hilbert space $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$. $\mathcal{H}$ has a basis $|N,n\rangle$ defined by $\hat{a}_1^\dagger|N,n\rangle=\sqrt{N+1}|N+1,n\rangle$, $\hat{a}_1|N,n\rangle=\sqrt{N}|N-1,n\rangle$,

$\hat{a}_2^\dagger|N,n\rangle = \sqrt{n+1}|N,n+1\rangle$ and $\hat{a}_2|N,n\rangle = \sqrt{n}|N,n-1\rangle$. This is just the tensor product of the harmonic oscillator eigenstates in each Hilbert space. The soliton described above, corresponding to a codimension-2 brane with $z_1 = 0$, is now described by $\hat{\Phi} = \lambda\hat{P}_1$, where $\hat{P}_1$ is given by

$$\hat{P}_1 = \sum_{n=0}^{\infty} |0,n\rangle\langle 0,n| \ . \tag{3.13}$$

The codimension-2 brane with $z_2 = 0$ is similarly given by $\hat{\Phi} = \lambda\hat{P}_2$,

$$\hat{P}_2 = \sum_{N=0}^{\infty} |N,0\rangle\langle N,0| \ . \tag{3.14}$$

3.1.2 Tachyon condensation

We will briefly review the idea of Sen [106], that D-branes are interpreted as solitons in the string tachyon field.

The bosonic D25-brane has a tachyon living on its world volume. The mass of the tachyon is $m^2_{tachyon} = -1/\alpha'$, and the potential is shown schematically in figure 3.1. It has been suggested [106] that the stable minimum of the potential corresponds to a 'vacuum' theory – one in which the (unstable) D25-brane has decayed and thus in which there are no possible open string excitations. This requires the difference in energy between the unstable local maximum and the local minimum to be equal to the D25-brane tension.

The effective action for the tachyon is

$$S = T_{25}\int d^{26}x\sqrt{(}g)\left(\frac{1}{2}f(t)g^{\mu\nu}\partial_\mu t\partial_\nu t + \ldots - V(t)\right) \ , \tag{3.15}$$

where '...' denotes higher derivative terms.

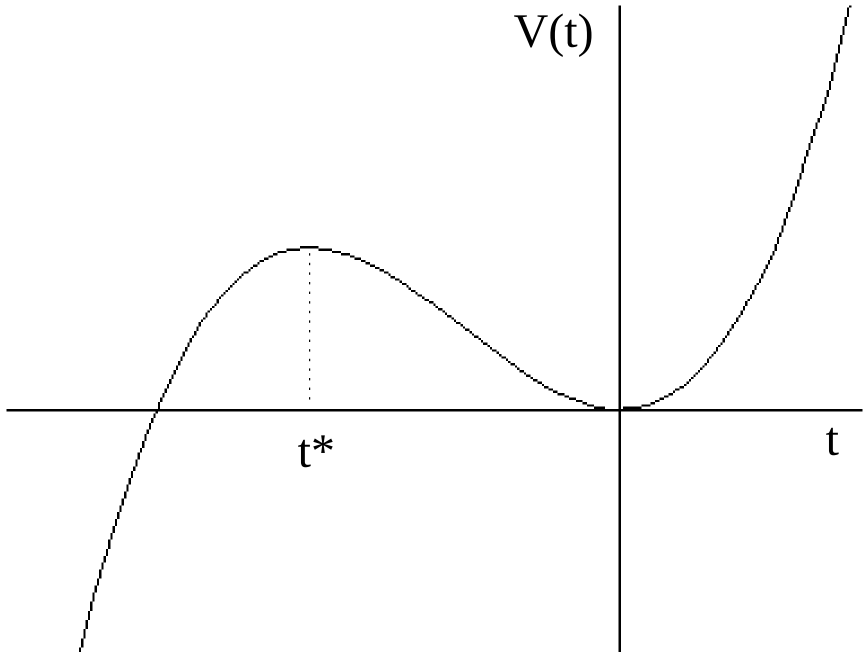


Figure 3.1: The tachyon potential in bosonic string theory.

With the potential V(t), it is possible to construct unstable solitons asymptotic to the closed string vacuum, *i.e.* for which asymptotically $t \to 0$, but $t \neq 0$ around some hyperplane. This soliton solution can be interpreted as a D-brane of lower dimension. These solitons have been studied in (truncated) string field theory and in boundary string field theory, but there many difficulties remain. In particular, the presence of higher derivatives in the kinetic energy terms in string field theory makes study there quite difficult.

3.1.3 Noncommutative tachyon and solitons

The study of tachyon condensation is greatly simplified in the presence of a strong magnetic field B. As described in the introductory chapter in section 1.4, the presence of magnetic field can be accounted for by replacing all regular products with the star product $\star$, while replacing the closed string metric g with the open string metric G.

Thus, one obtains the following action for the tachyon

$$S = T_{25} \int d^{26}x \sqrt(G) \left(\frac{1}{2} f(t) G^{\mu\nu} \partial_\mu t \partial_\nu t + \ldots - V(t) \right) , \tag{3.16}$$

where $\star$-products are implicit.

As described at the beginning of this section (for the particular case of four dimensions, but the idea is trivially extended to any number of dimensions), in the limit of large noncommutativity (large B-field), the kinetic energy terms, including the higher derivative terms, can be dropped and soliton solutions can be constructed. These should correspond to lower dimensional D-branes. In this set-up, it is possible to show that the solitons can be interpreted as D-branes, since they have the right properties, such as correct tensions and tachyon masses. [69].

3.2 Unitary fluctuations of single 2-brane

We will now explore the zero-energy fluctuations about the 2-brane solution presented in section 3.1. Consider the soliton given by $\hat{U}^\dagger \hat{P}_1 \hat{U}$, where $\hat{U}$ is some unitary operator on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. We are interested in the kinetic energy as a function of $\hat{U}$. This is more involved than before, so we will work only to second order with

$$\hat{U} = e^{i\epsilon\hat{\Lambda}} = 1 + i\epsilon\hat{\Lambda} + -\frac{1}{2}\epsilon^2\hat{\Lambda}^2 + \mathcal{O}(\epsilon^3) \tag{3.17}$$

for ϵ real and small and $\hat{\Lambda}$ Hermitian. Define

$$\hat{\Lambda}|0, j\rangle = \sum_{I,i} b^j_{Ii} |I, i\rangle . \tag{3.18}$$

Following [58], we now obtain the effective Hamiltonian for small fluctuations of the brane. In the operator language, the kinetic energy is:

$$\begin{aligned} K &= -\frac{1}{2\theta^2}\sum_{k=1}^{2}\mathrm{tr}\{[\hat{x}_k,\hat{\Phi}]^2+[\hat{y}_k,\hat{\Phi}]^2\} = \frac{1}{\theta^2}\sum_{k=1}^{2}\mathrm{tr}\{[\hat{a}_k,\hat{\Phi}][\hat{\Phi},\hat{a}_k^\dagger]\} \\ &= \frac{2}{\theta}\mathrm{tr}\{\hat{\Phi}\hat{H}\hat{\Phi}-\sum_{k=1}^{2}(\hat{\Phi}\hat{a}_k\hat{\Phi})(\hat{\Phi}\hat{a}_k^\dagger\hat{\Phi})\}\ , \end{aligned} \tag{3.19}$$

where $\hat{H}$ is the harmonic oscillator Hamiltonian,

$$\hat{H} \equiv \sum_{k=1}^{2}\left(\hat{a}_k^\dagger\hat{a}_k+\frac{1}{2}\right)\ . \tag{3.20}$$

Any projection operator, $\hat{A}$, such as our soliton, projects onto a subspace, $\mathcal{H}_A$, of the Hilbert space $\mathcal{H}$. Let $|i\rangle$, $i\in\mathcal{S}$, be a basis for $\mathcal{H}_A$. Then we can write the kinetic energy as

$$K=\frac{\lambda^2}{\theta^2}\left(\sum_{i\in\mathcal{S};k=1,2}\langle i|\hat{a}_k^\dagger\hat{a}_k+\hat{a}_k\hat{a}_k^\dagger|i\rangle-2\sum_{i,j\in\mathcal{S};k=1,2}|\langle i|\hat{a}_k^\dagger|j\rangle|^2\right)\ . \tag{3.21}$$

This form is sometimes more useful for calculation.

For fluctuations about P_1, to second order in ϵ we obtain:

$$\begin{aligned} \frac{\theta}{2\lambda^2}K &= \sum_k(2k+2)+2\epsilon^2\Bigg[\sum_{I\geq 2,j,k\geq 0}(I+k-j)|b_{Ii}^k|^2-\sum_{j,k\geq 0}|b_{1j}^k|^2 \\ &- \sum_{j\geq 0}(j+1)\left(1-\sum_{I\geq 1,i\geq 0}|b_{Ii}^j|^2-\sum_{I\geq 1,i\geq 0}|b_{Ii}^{j+1}|^2\right) \\ &- \sum_{I,i,j\geq 1}\sqrt{i(j+1)}\left(\bar{b}_{I,i-1}^j b_{I,i}^{j+1}+b_{I,i-1}^j\bar{b}_{I,i}^{j+1}\right)\Bigg]\ . \end{aligned} \tag{3.22}$$

This can be rearranged to the positive definite form:

$$\frac{\theta}{2\lambda^2}K = T + 2\epsilon^2 \left[\sum_{I\geq 2;i,k\geq 0} I|b^k_{Ii}|^2 + \sum_{I\geq 1;i\geq 0} i|b^0_{Ii}|^2 \right.$$
$$\left. + \sum_{I\geq 1;j,k\geq 0} \left| \sqrt{k+1} b^k_{Ij} - \sqrt{j+1} b^{k+1}_{I,j+1} \right|^2 \right] . \tag{3.23}$$

Here, T is an infinite constant corresponding to the zero-point energy of the infinite brane. The massless modes must satisfy

$$b^m_{Ii} = 0 \,(\text{for } I \geq 2),$$
$$b^0_{1,i} = 0 \,(\text{for } i \geq 1), \tag{3.24}$$
$$\sqrt{m+1} b^m_{1,n} = \sqrt{n+1} b^{m+1}_{1,n+1} . \tag{3.25}$$

The solution to these constraints is

$$b^m_{1,n} = \begin{cases} 0 & \text{for } m < n \\ \sqrt{\frac{m!}{n!}} c_{m-n} & \text{for } m \geq n \end{cases} , \tag{3.26}$$

where c_m $(m = 0, 1, \ldots)$ are arbitrary constants. Note that when looking at the original form of the kinetic energy (3.23), we are cancelling two divergent sums. If we demand that all sums converge, the following solution is not legitimate. Throwing caution to the wind, we define the entire holomorphic function $f(\zeta) = \sum_m c_m \zeta^m$. $\hat{\Lambda}$ can then be written as:

$$\hat{\Lambda} = \hat{a}_1^\dagger f(\hat{a}_2) + \hat{a}_1 f(\hat{a}_2)^\dagger + \mathcal{O}(\epsilon^2) , \tag{3.27}$$

and the transformed soliton is:

$$\hat{\Phi} = \lambda \hat{U}^\dagger \hat{P}_1 \hat{U}, \qquad \hat{U} = e^{i\epsilon(\hat{a}_1^\dagger f(\hat{a}_2) + \hat{a}_1 f(\hat{a}_2)^\dagger)} + \mathcal{O}(\epsilon^2)\ . \tag{3.28}$$

Physically, this is interpreted as a deformation of the brane from $z_1 = 0$ to $z_1 = \sqrt{2\theta}\epsilon f(\frac{z_2}{\sqrt{2\theta}})$. We can now understand the divergences in this solution as stemming from the fact that a nonconstant entire function cannot be bounded and, as such, these are infinitely large deformations of the brane. If we cut off the sums to force them to be finite, we can still understand these as local approximate zero modes.

We can rearrange the terms in the kinetic energy into the following (also positive definite) form:

$$\frac{\theta}{2\lambda^2} K = T + 2\epsilon^2 \Bigg[\sum_{I\geq 2, i, k\geq 0} I |b^k_{Ii}|^2 + \sum_{I\geq 1, k\geq 0} k |b^k_{I0}|^2 + \sum_{I\geq 1, j, k\geq 0} \left| \sqrt{j+1} b^k_{Ij} - \sqrt{k+1} b^{k+1}_{I,j+1} \right|^2 \Bigg] . \tag{3.29}$$

Repeating the above analysis, we find that the massless modes for this form of the kinetic energy are

$$b^m_{1,n} = \begin{cases} 0 & \text{for } m > n \\ \sqrt{\frac{n!}{m!}} c_{n-m} & \text{for } m \leq n \end{cases} . \tag{3.30}$$

Taking again $f(\zeta) = \sum_m c_m \zeta^m$, we obtain

$$\Lambda = z_1^\dagger f(z_2^\dagger) - z_1 f(z_2^\dagger)^\dagger\ . \tag{3.31}$$

This corresponds to a deformation of the brane from $z_1 = 0$ to $z_1 = \sqrt{2\theta}\epsilon f(\frac{\bar{z}_2}{\sqrt{2\theta}})$, an antiholomorphic deformation.

3.3 Classical Geometry

We will consider surfaces in $\mathbb{R}^4$ that can be described by a holomorphic equation when $\mathbb{R}^4$ is identified with $\mathbb{C}^2$. Such surfaces have a minimal area in the sense that small deformations of the surface, keeping the boundary conditions at infinity intact, never decrease the area. Let the coordinates be:

$$z_k \equiv x_k + iy_k, \qquad k = 1, 2 \ . \tag{3.32}$$

Consider first a surface that spans the z_2-direction and is given by the equation $z_1 = 0$. Small holomorphic deformations are described by $z_1 = \epsilon f(z_2)$ with $f(z) = \sum_{n=0}^{\infty} c_n z^n$ a holomorphic function. These are the fluctuations we found in the previous section. Of course, antiholomorphic fluctuations are also allowed.

Now consider adding a second surface spanning the z_1 direction, with the equation $z_2 = 0$. The two surfaces can be represented together by the equation $z_1 z_2 = 0$. This reducible surface can be deformed into a smooth irreducible surface given by $z_1 z_2 = \zeta$ where ζ is a complex number. This is the only holomorphic deformation of the singular surface $z_1 z_2 = 0$ that preserves the boundary conditions $z_1 \to 0$ as $|z_2| \to \infty$ and $z_2 \to 0$ as $|z_1| \to \infty$.

In this case, we see that the possible deformations are given by $z_1 = \epsilon f(z_2)$ where $f(z) = \sum_{n=-1}^{\infty} c_n z^n$ is allowed to have a simple pole at $z = 0$. More generally, if we add r surfaces given by the planes $z_2 = \xi_j$ $(j = 1 \ldots r)$, we can have deformations $z_1 = \epsilon f(z_2)$ where f is a meromorphic function that is allowed to have simple poles at $\xi_1, \ldots, \xi_r$. If we add a surface $z_2 = 0$ with multiplicity k, then $f(z)$ is allowed to have a pole of k^{th} order at the origin.

3.4 Intersecting D2-Branes

3.4.1 Construction of the Intersecting Soliton

In the previous section, we constructed a D2-brane at $z_1 = 0$ as $\hat{\Phi}_1 = \lambda \hat{P}_1$ and a D2-brane at $z_2 = 0$ as $\hat{\Phi}_2 = \lambda \hat{P}_2$. We now wish to find a soliton $\hat{\Phi} = \lambda \hat{P}$ which asymptotically looks like $\hat{\Phi}_1 + \hat{\Phi}_2$. This is straightforward. We define

$$\hat{P}_\eta = \hat{P}_1 + \hat{P}_2 - \eta \hat{P}_1 \hat{P}_2, \qquad \hat{\Phi}_\eta = \lambda \hat{P}_\eta \ . \tag{3.33}$$

This will be a projection operator for $\eta = 1$ or $\eta = 2$. To distinguish between the two solutions, we need to calculate their kinetic energy, (3.21). While each solution has an infinite kinetic energy because of its infinite extent, the difference is finite and easy to calculate:

$$K(\hat{\Phi}_{\eta=2}) - K(\hat{\Phi}_{\eta=1}) = \frac{4\lambda^2}{\theta} \ . \tag{3.34}$$

Thus, $\eta = 1$ corresponds to the solution with the lower kinetic energy. We propose that this solution corresponds to two intersecting branes. The $\eta = 2$ solution is similar, but it has a 'hole' attached at the intersection:

$$\hat{P}_{\eta=2} = \hat{P}_{\eta=1} - \hat{P}_1 \hat{P}_2 \ . \tag{3.35}$$

In a sense, it is as if a 0-brane (represented by $\hat{P}_1 \hat{P}_2$) had been removed. This solution will turn out to be unstable to small unitary perturbations.

3.4.2 Fluctuations

We now wish to repeat the calculation of the effective Hamiltonian for small fluctuations of the two intersecting branes. Consider the fluctuation given by $\hat{U}^\dagger \hat{P}_\eta \hat{U}$, where

$\hat{U}$ is again a unitary operator on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. As before, let

$$\hat{U} = e^{i\epsilon\hat{\Lambda}} = 1 + i\epsilon\hat{\Lambda} - \frac{1}{2}\epsilon^2\hat{\Lambda}^2 + \mathcal{O}(\epsilon^3) \ , \tag{3.36}$$

with ϵ real and small and $\hat{\Lambda}$ hermitian. One can calculate the kinetic energy for this soliton to second order in ϵ. This is most conveniently done from equation (3.21).

In the $\eta = 1$ case, we define

$$\begin{aligned} \hat{\Lambda}|0,j\rangle &= \sum_{I,i} b^j_{Ii}|I,i\rangle \ , \quad I,i,j \geq 1 \ , \\ \hat{\Lambda}|J,0\rangle &= \sum_{I,i} c^J_{Ii}|I,i\rangle \ , \quad I,i,J \geq 1 \ , \\ \hat{\Lambda}|0,0\rangle &= \sum_{I,i} d_{Ii}|I,i\rangle \ . \quad I,i \geq 1 \ . \end{aligned} \tag{3.37}$$

After consolidation of terms, (3.21) becomes:

$$\begin{aligned} \frac{\theta}{2\lambda^2}K_{\eta=1} &= \frac{\theta}{2\lambda^2}K(\hat{\Phi}_{\eta=1}) + 2\epsilon^2\Bigg[\sum_{J\geq 2,j,k\geq 1} J|b^k_{Jj}|^2 + \sum_{j\geq 2,J,K\geq 1} j|c^K_{Jj}|^2 \\ &+ \sum_{J,j,k\geq 1}\left|\sqrt{k+1}b^k_{Jj} - \sqrt{j+1}b^{k+1}_{J,j+1}\right|^2 + \sum_{J,j,K\geq 1}\left|\sqrt{K+1}c^k_{Jj} - \sqrt{J+1}c^{K+1}_{J+1,j}\right|^2 \\ &+ \sum_{J\geq 2,j\geq 1} J|d_{Jj}|^2 + \sum_{j\geq 2,J\geq 1} j|d_{Jj}|^2 \\ &+ \sum_{J,j\geq 1}\left|d_{Jj} - \sqrt{j+1}b^1_{J,j+1}\right|^2 + \sum_{J,j\geq 1}\left|d_{Jj} - \sqrt{J+1}c^1_{J+1,j}\right|^2\Bigg] \ , \end{aligned} \tag{3.38}$$

where $K(\hat{\Phi}_{\eta=1})$ is the (infinite) energy of an undistorted soliton discussed in previous subsection.

Using the same procedure as before, we obtain the following zero modes:

$$b_{1,n}^{m} = \begin{cases} 0 & \text{for } m+1<n \\ \frac{d_{11}}{\sqrt{n}} & \text{for } m+1=n \\ d_{10} & \text{for } m=n \\ \sqrt{\frac{m!}{n!}}p_{m-n} & \text{for } m>n \end{cases} \tag{3.39}$$

and

$$c_{N,1}^{M} = \begin{cases} 0 & \text{for } M+1<N \\ \frac{d_{11}}{\sqrt{N}} & \text{for } M+1=N \\ d_{01} & \text{for } M=N \\ \sqrt{\frac{M!}{N!}}q_{M-N} & \text{for } M>N \end{cases} , \tag{3.40}$$

with d_{Jj} for all $J, j \geq 2$ equal to zero. The p's and q's are arbitrary constants. These can be used to define two entire holomorphic functions $f_1(\zeta) = \sum_m p_m \zeta^m$ and $f_2(\zeta) = \sum_M q_M \zeta^M$. These zero modes, just as for a single brane, correspond to deformations of the two branes: $z_1' = \epsilon f_1(z_2)$ and $z_2' = \epsilon f_2(z_1)$. As in the case of a single brane, the terms in the kinetic energy can be rearranged to make apparent the antiholomorphic deformations.

A new phenomenon is the mode corresponding to a non-zero d_{11} together with $b_{1,k+1}^{k} = d_{11}(k+1)^{-1/2}$ and $c_{K+1,1}^{K} = d_{11}(K+1)^{-1/2}$ so that the terms in kinetic energy that are differences vanish. This mode might be thought of as

$$\Lambda \sim \frac{\alpha}{z_1 z_2} + \frac{\bar{\alpha}}{z_1^\dagger z_2^\dagger} \ . \tag{3.41}$$

This is a complex mode (two real modes) corresponding to the extra degrees of freedom living on the intersection of the two branes.

We now consider the case of $\eta = 2$. Here, (3.21) reduces to

$$\begin{aligned}
\frac{\theta}{2\lambda^2} K_{\eta=1} &= \frac{\theta}{2\lambda^2} K(\hat{\Phi}_{\eta=1}) + 2\epsilon^2 \Bigg[\sum_{J\geq 2, j, k\geq 1} J |b_{Jj}^k|^2 + \sum_{j\geq 2, J, K\geq 1} j |c_{Jj}^K|^2 \\
&+ \sum_{J,j,k\geq 1} \left| \sqrt{k+1} b_{Jj}^k - \sqrt{j+1} b_{J,j+1}^{k+1} \right|^2 + \sum_{J,j,K\geq 1} \left| \sqrt{K+1} c_{Jj}^k - \sqrt{J+1} c_{J+1,j}^{K+1} \right|^2 \\
&+ \sum_{J,k\geq 2} (J+k-1) |b_{Jk}^1|^2 + \sum_{J\geq 2} (J-1) |b_{J+1,1}^1|^2 + \sum_{k\geq 2} (k-1) |b_{1,k}^1|^2 \\
&+ \sum_{j,K\geq 2} (j+K-1) |c_{Kj}^1|^2 + \sum_{j\geq 2} (j-1) |c_{1,j+1}^1|^2 + \sum_{K\geq 2} (K-1) |c_{K,1}^1|^2 \\
&+ \sum_{k\geq 2} k |b_{00}^{k+1}|^2 + \sum_{K\geq 2} K |c_{00}^{K+1}|^2 + \sum_{K\geq 2} |b_{K1}^1 + \bar{c}_{00}^K|^2 + \sum_{k\geq 2} |c_{1k}^1 + \bar{b}_{00}^k|^2 \\
&- \left| b_{11}^1 + \bar{c}_{00}^1 \right|^2 - \left| c_{11}^1 + \bar{b}_{00}^1 \right|^2 \Bigg] . \qquad (3.42)
\end{aligned}$$

The zero modes, which we will not write out explicitly, include our familiar entire holomorphic and anti-holomorphic deformations of the branes. More importantly, we now have unstable modes given by $b_{11}^1 + \bar{c}_{00}^1 \neq 0$ and $c_{11}^1 + \bar{b}_{00}^1 \neq 0$ together with constraints setting the terms that are differences to zero. These two modes correspond to moving the aforementioned 'hole' away from the intersection along either of the two branes. We also note that the above effective Hamiltonian has an additional zero mode given by $\Lambda = \alpha(\hat{a}_1^\dagger)^2 + \bar{\alpha}(\hat{a}_1)^2$ (and similarly for $\hat{a}_2$), which corresponds to distorting the shape of the hole from the Gaussian ground state of a harmonic oscillator into a squeezed state.

3.5 Extension to Multiple Branes

Our construction for two intersecting D2-branes can easily be extended to a larger number of branes.

For example, let $\hat{P}_1^K$ be a projection operator corresponding to a stack of K branes at $z_1 = 0$ and $\hat{P}_2^L$ be a projection operator corresponding to a stack of L branes at $z_2 = 0$. This means that $\hat{P}_1^K$ can be written as a sum of K projection operators

$$\hat{P}_1^K = \sum_{i=1}^{K} \hat{p}_1^i \tag{3.43}$$

with $\hat{p}_1^i \hat{p}_1^j = \delta^{ij} \hat{p}_1^i$, each $\hat{p}_1^i$ being a projection operator for a single brane. Similarly,

$$\hat{P}_2^L = \sum_{i=1}^{L} \hat{p}_2^i \ . \tag{3.44}$$

Now, any operator of the form

$$\hat{P}_1^K + \hat{P}_2^L - \sum_{i=1}^{K} \sum_{j=1}^{L} \eta_{ij} \hat{p}_1^i \hat{p}_2^j \tag{3.45}$$

for $\eta_{ij} = 1, 2$ corresponds to an intersection of these two stacks.

As another example, let us take $\mathbb{R}^6$, i.e. three complex dimensions. Let $\hat{P}_{12}$ correspond to a codimension-2 brane at $z_3 = 0$, $\hat{P}_{23}$ correspond to a codimension-2 brane at $z_1 = 0$ and $\hat{P}_{31}$ correspond to a codimension-2 brane at $z_2 = 0$. Then it can be checked that

$$\begin{aligned} \hat{P}_{12} + \hat{P}_{23} + \hat{P}_{31} - \eta_{12} \hat{P}_{23} \hat{P}_{31} - \eta_{23} \hat{P}_{12} \hat{P}_{31} - \eta_{31} \hat{P}_{12} \hat{P}_{23} \\ +(\eta_{12} + \eta_{23} + \eta_{31} - \eta - 1) \hat{P}_{12} \hat{P}_{23} \hat{P}_{31} \end{aligned} \tag{3.46}$$

is a projection operator corresponding to the intersection of all three branes at a point, provided we set $\eta_{12}, \eta_{23}, \eta_{31}, \eta \in \{1, 2\}$. It is straightforward, if a bit tedious, to extend this to any number of branes.

Chapter 4

Gravity duals of nonlocal field theories

The gravity duals of nonlocal field theories in the large N limit exhibit a novel behavior near the boundary. In this chapter, we present and study the gravity duals of dipole theories – a particular class of nonlocal theories with fundamental dipole fields. The nonlocal interactions are manifest in the metric of the gravity dual. We compare the situation to that in noncommutative SYM.

The chapter begins with a brief introduction to the most studied gravity dual – the AdS-CFT correspondence, followed by an introduction to its extension to noncommutative SYM.

4.1 Introduction

The AdS-CFT correspondence is a remarkable idea that type IIB superstring theory on a particular background ($\mathbf{AdS}_5 \times \mathbf{S}^5$) is dual, or equivalent, to a conformal field theory, the $\mathcal{N} = 4$ SYM. The idea was originally proposed in [83, 63, 121]. For introductory reviews, see for example [5, 78].

$\mathcal{N} = 4$ U(N) SYM theory lives on the world-volume of N coincident D3-branes. For large N and maximal supersymmetry, these branes are a heavy object in type IIB supergravity, with a metric

$$ds^2 = H(r)^{-\frac{1}{2}}\left(dt^2 - dx_1^2 - dx_2^2 - dx_3^2\right) - H(r)^{\frac{1}{2}}\left(dr^2 + r^2 d\Omega_5^2\right) \ , \tag{4.1}$$

where

$$H(r) = 1 + \left(\frac{\mathcal{R}}{r}\right)^4 \ , \tag{4.2}$$

and a constant dilaton background $\phi = \phi_0$. The geometry of this solution is such that the branes are at the end of a semi-infinite throat at $r \to 0$. For large r, away from the branes, the space is flat. For $r \ll \mathcal{R}$, this metric becomes

$$ds^2 = \left[\left(\frac{r}{\mathcal{R}}\right)^2\left(dt^2 - dx_1^2 - dx_2^2 - dx_3^2\right) - \left(\frac{\mathcal{R}}{r}\right)^2 dr^2\right] - \mathcal{R}^2 d\Omega_5^2 \ , \tag{4.3}$$

which is just the metric for $\mathbf{AdS}_5 \times \mathbf{S}^5$ of equal radii, $\mathcal{R}$. We can rewrite this metric in terms of $u = \mathcal{R}^2/r$,

$$ds^2 = \left[\left(\frac{\mathcal{R}}{u}\right)^2\left(dt^2 - dx_1^2 - dx_2^2 - dx_3^2 - du^2\right)\right] - \mathcal{R}^2 d\Omega_5^2 \ . \tag{4.4}$$

At the boundary, which is located at $u \to 0$, $r \to \infty$, the metric of $\mathbf{AdS}_5$ becomes infinite. The boundary is described by classical geometry, and quantum gravity on $\mathbf{AdS}_5$ corresponds to a local field theory on a classical space [84, 63, 121, 5].

The AdS-CFT conjecture is that type IIB string theory on the background given by (4.4) is equivalent to the conformal field theory living on the worldvolume of the stack of D3-branes, namely $\mathcal{N} = 4$ U(N) SYM. There are several different ways of justifying this conjecture. A simple way to think about it is that there are two different but equivalent descriptions of a stack of D3-branes. One description is via the metric (4.2). In this description, the branes themselves are described by the throat region $r \to 0$, *i.e.* by the metric (4.4). The other, equivalent, description is as an effective theory living on the branes themselves, which is the CFT.

What is the precise statement of correspondence between these two theories? Consider any field living in the bulk, and the corresponding operator in the CFT living on the branes, which acts as a source for the bulk field. For example, from the Born-Infeld action, we have an interaction term [78]

$$S_{int} = \frac{\sqrt{\pi}}{\kappa} \int d^4x \; tr \left(\frac{1}{4} \phi F^2_{\alpha\beta} \right) \tag{4.5}$$

between the dilaton and trF^2. This means that a dilaton can be converted into a pair of gluons on the worldvolume. In general, we have a coupling between a bulk field $\phi(x, u)$ and an operator $\mathcal{O}(x)$ in the CFT,

$$S_{int} = \int d^4x \; \phi(x, 0) \mathcal{O}(x) \; . \tag{4.6}$$

The generating functional of the connected correlation functions of $\mathcal{O}$ in the CFT is conjectured to be equal to the extremum of the classical string theory action I, subject to constraints that the bulk field $\phi(x, u) = \phi_0(x)$ at $u = \mathcal{R}$

$$W[\phi_0(x)] \equiv \langle e^{\int d^4x \;\; \phi_0(x) \mathcal{O}(x)} \rangle \;\; = \;\; I|_{\phi(x, u=\mathcal{R}) = \phi_0(x)} \; . \tag{4.7}$$

The exact point at which the boundary conditions are imposed on the string theory side is not important, since the theory is conformally invariant and so any $u = u_{\text{cutoff}}$ will do. One can think of u_{cutoff} as simply the UV regulator for the theory, since the radial coordinate of $\mathbf{AdS}_5$ can be thought of as an energy scale of the gauge theory, with $u \to 0$ being the UV region.

We mentioned above that since the metric of $\mathbf{AdS}_5$ becomes infinite at the boundary, the boundary is described by classical geometry, and quantum gravity on $\mathbf{AdS}_5$ corresponds to a *local* field theory on a classical space. The AdS/CFT correspondence can be extended to field theories on a noncommutative space [72, 84, 7]. The gravity dual for the large N limit of $\mathcal{N} = 4$ SYM theory on a noncommutative $\mathbb{R}^4$ (NCSYM) with the noncommutativity in the $2, 3$ directions has the metric [72, 84]

$$ds^2 = \frac{\mathcal{R}^2}{u^2}(dt^2 - dx_1^2 - h(dx_2^2 + dx_3^2) - du^2) \ , \tag{4.8}$$

where $h = \frac{u^4}{u^4+\theta^2}$ with $\theta = \theta_{23}$ the determining typical length scale in the theory. Here the boundary, $u \to 0$, is no longer classical. Indeed some components of the metric tend to zero on the boundary.

Our motivation in this chapter is to understand how nonlocality in the field theory affects the metric of its gravity dual near the boundary. Unfortunately, field theories on noncommutative spaces can be quite complicated; they exhibit UV/IR mixing and nonlocal behavior on varying scales. UV/IR mixing, which means that high momentum is associated with large-scale nonlocality and arbitrarily small momentum introduces a new short-distance scale, can even obstruct the renormalization procedure [86, 116]. Although $\mathcal{N} = 4$ NCSYM is a finite theory and renormalizability is not an issue, noncommutative geometry doesn't appear to be the simplest way to introduce nonlocality. There is a simpler way.

We study a class of nonlocal gauge field theories in which some of the fields correspond to dipoles of a constant length. Such theories were discussed in [17] in the context of T-duality in noncommutative geometry. They were realized in string theory in a different setting in [37]. Also see [122, 29, 30] for previous appearances of such theories.

At low energies these "dipole theories" can be described as a deformation of $\mathcal{N} = 4$ SYM by a vector operator of conformal dimension 5. This can be compared to the deformation by a tensor operator of conformal dimension 6 that describes NCSYM at low energy [104, 81, 54]. If the conformal dimension and the size of the Lorentz representation is an indication of simplicity, then it is reasonable to expect that dipole theories might be simpler than NCSYM.

The more interesting questions, however, hover in the UV region of the theory. At distances shorter than the scale of the nonlocality, we expect to find new phenomena.

Our ultimate goal is to answer the following questions:

- How is the nonlocality of the dipole theory manifested in the boundary metric?
- How does this manifestation of nonlocality compare to that of noncommutative geometry? Are these features generic to the gravity duals of nonlocal field theories?

In section 4.6 we answer the first question for a particularly simple dipole theory. In section 4.9, we show that this effect is analogous to a feature of the supergravity dual of noncommutative geometry. We also make some comments about the nature of the supergravity dual for generic nonlocal theories.

The particular dipole theory that we study breaks supersymmetry entirely. We chose to work with it because the supergravity equations are simplified. The fermionic degrees of freedom, however, require extra care. In fact, type-0 string theory with a strong RR field strength has to be used in order to correctly describe the gravity dual [16].

This chapter is organized as follows. In section 4.2 we review the construction of dipole theories. In section 4.3 we describe a simple string theory realization of these theories and then calculate their gravity dual in section 4.4. In section 4.5 we study the geometry of the gravity dual. In section 4.6 we demonstrate the nonlocality of the boundary. In section 4.7 we compute some correlation functions and show how they exhibit some generic features of nonlocality. In section 4.8 we compute the gravity dual for generic dipole theories. Finally, in section 4.9 we discuss how the features we have found here might be generic to the supergravity duals of all nonlocal field theories.

4.2 Dipole Theories

Dipole theories are nonlocal field theories that also break Lorentz invariance. They were obtained in [17] by studying the T-duals of twisted fields in noncommutative gauge theory. Below, we will describe how to make a dipole theory out of an ordinary field theory.

4.2.1 Definition

We start with a local and Lorentz invariant field theory in d dimensions. In order to turn it into a nonlocal theory we assign to every field Φ_a a vector L_a^μ ($\mu = 1 \dots d$). We will call this the "dipole vector" of the field.

The fields Φ_a can be scalars, fermions, or have higher spin. Next, we define a noncommutative product

$$(\Phi_1 \widetilde{\star} \Phi_2)_x \equiv \Phi_1(x - \frac{1}{2}L_2)\Phi_2(x + \frac{1}{2}L_1) \ . \tag{4.9}$$

It is easy to check that this defines an associative product provided that the vector assignment is additive, that is, $\Phi_1 \widetilde{\star} \Phi_2$ is assigned the dipole vector $L_1 + L_2$. For CPT symmetry, we will require that if Φ has dipole vector L then the charge conjugate field, $\Phi^\dagger$, is assigned the dipole vector $-L$. We will also require that gauge fields have zero dipole length.

In order to construct the Lagrangian of the dipole theory we need to replace the ordinary product of fields with the noncommutative $\widetilde{\star}$-product (4.9). In general, there might be some ordering ambiguity, but the theories we will consider below are $SU(N)$ gauge theories and have a natural ordering induced from the noncommutative products of $N \times N$ matrices.

We have seen that the requirement of associativity translates into a requirement of additivity for the dipole vectors. One way to ensure this is to have a global conserved charge in the theory such that a field Φ_a has charge Q_a. We then pick a constant vector L^μ and assign to every field Φ_a ($a = 1 \ldots n$, where n is the number of fields in the theory) the dipole vector $Q_a L^\mu$. More generally, we can have m global charges such that a field Φ_a has the charges Q_{ja} ($j = 1 \ldots m$). We can then pick a constant $d \times m$ matrix $\Theta^{\mu j}$ ($\mu = 1 \ldots d$ and $j = 1 \ldots m$) and assign the field Φ_a a dipole vector $\sum_{j=1}^m \Theta^{\mu j} Q_{ja}$.

Extending this definition by allowing Q_a to be the momentum we see that non-commutative Yang-Mills theory can also be thought of as a dipole theory. The matrix $\Theta^{\mu j}$ then becomes $\Theta^{\mu\nu}$ ($\nu = 1 \ldots d$) and is required to be antisymmetric. The dipole lengths are then both proportional to and transverse to the momentum [23, 123, 107].

4.2.2 A Dipole Deformation of $\mathcal{N} = 4$ SYM

The dipole theories that we study in the rest of this chapter can be obtained from ordinary $SU(N)$ $\mathcal{N} = 4$ SYM in 3+1D by turning the scalars and fermions into dipole fields. $\mathcal{N} = 4$ SYM has 6 real scalars in the representation **6** of the R-symmetry group

$SU(4)$ and 4 Weyl fermions in the representation **4** of $SU(4)$. We will use the global R-symmetry charges to determine the dipole vectors of the various fields as follows. Pick 3 constant commuting elements $V^\mu \in su(4)$ ($\mu = 1 \ldots 3$ and we will not consider time-like dipole vectors in this chapter), where $su(4)$ is the Lie algebra of $SU(4)$. Take V^μ to have dimensions of length. Denote the matrix elements of V^μ in the representation **4** as $\hat{U}^\mu_{j\overline{k}}$ ($j, \overline{k} = 1 \ldots 4$). Here $\hat{U}^\mu$ is a traceless Hermitian 4×4 matrix. Denote the matrix elements of V^μ in the representation **6** as M^μ_{ab} ($a, b = 1 \ldots 6$). M^μ is a real antisymmetric 6×6 matrix.

Let $u^{(l)}_a$ ($a, l = 1 \ldots 6$) be an eigenvector of M^μ with (real) eigenvalue $\tilde{L}^\mu_l$ so that $\sum_b M^\mu_{ab} u^{(l)}_b = \tilde{L}^\mu_l u^{(l)}_a$. $u^{(l)}_a$ does not depend on μ because $[M^\mu, M^\nu] = 0$. Let ϕ_a ($a = 1 \ldots 6$) be the 6 real scalar fields of $\mathcal{N} = 4$ SYM. Then the complex valued scalar fields $\phi^{(l)} \equiv \sum_a u^{(l)}_a \phi_a$ are assigned a dipole vector with components $2\pi \tilde{L}^\mu_l$ ($\mu = 1 \ldots d$). Similarly, the fermionic fields are assigned dipole vectors that are determined by the eigenvalues of the matrices $\hat{U}^\mu$.

4.2.3 Supersymmetry

The dipole theories obtained from $\mathcal{N} = 4$ SYM in the previous subsection are parameterized by d constant traceless Hermitian 4×4 matrices $\hat{U}^\mu$. For simplicity we will set $\hat{U}^1 = \hat{U}^2 = 0$ and $\hat{U} \equiv \hat{U}^3$. Thus, the dipole vectors are all in the 3^{rd} direction. The matrix $\hat{U}$ has dimensions of length, and its eigenvalues determine the dipole vectors of the various fields. Let the eigenvalues be $\alpha_1, \alpha_2, \alpha_3, -(\alpha_1 + \alpha_2 + \alpha_3)$. Then, the dipole vectors of the various scalar fields are given by $\pm(\alpha_i + \alpha_j)$ ($1 \le i < j \le 3$).

The number of supersymmetries that are preserved by the dipole theory is determined by the rank r of $\hat{U}$:

- If $r = 4$, then the theory is not supersymmetric at all.

- If $r = 3$, there is one zero eigenvalue that we take by convention to be $\alpha_3 = 0$, and the theory has $\mathcal{N} = 1$ supersymmetry.

- If $r = 2$, there are two zero eigenvalues that we take to be $\alpha_2 = \alpha_3 = 0$. The theory then has $\mathcal{N} = 2$ supersymmetry. The vector multiplet of $\mathcal{N} = 4$ SYM decomposes as a vector multiplet and a hypermultiplet of $\mathcal{N} = 2$ SYM. All the fields in the $\mathcal{N} = 2$ vector multiplet have dipole vector 0, and the fields in the hypermultiplet have dipole vectors $\pm\alpha_1$.

Because we can realize dipole theories without supersymmetry, one might ask if poles similar to those discovered in [86, 116] might arise in the perturbative expansion of the theory. In fact, they do not. This can be seen by examining the expression of [86, 116] for the effective cutoff

$$\Lambda_{\text{eff}} \to \frac{1}{\sqrt{\Lambda^{-2} + (\theta p)^2}} \ . \tag{4.10}$$

We recognize θp as the length of the dipoles in noncommutative geometry. Thus, the analogous expression in our theory is

$$\Lambda_{\text{eff}} \to \frac{1}{\sqrt{\Lambda^{-2} + L^2}} \ , \tag{4.11}$$

which, as it is independent of the momenta, gives rise to no new poles.

4.3 String Theory Realization of Dipole Theories

In order to find the gravity dual of the large N limit of a particular dipole theory, we need to find a simple string theory realization for it. We now do this for a large class of dipole theories.

In [37], a realization of dipole theories with $\mathcal{N} = 2$ supersymmetry was suggested using D3-branes that probe the center of a modified Taub-NUT geometry. While this realization is convenient for a BPS analysis it is hard to extract the gravity dual from it, and it is not obvious how to generalize it to dipole theories that break $\mathcal{N} = 2$ supersymmetry.

Fortunately, the Taub-NUT space that was used in [37] is not an essential ingredient. We can find an alternative setting that has the same behavior near the brane probes. This setting, which we will describe below, has the disadvantage that the geometry is not asymptotically Euclidean at infinity. Nevertheless, it has been constructed in string theory [99] and is good for extracting the gravity duals that we seek. Other worldsheet CFTs that break Lorentz invariance have been studied in [39].

The backgrounds that we consider are twisted versions of type-II string theory. They are related to the Melvin solution [85] and are in fact identical to the backgrounds discussed in [48] and more recently in [36, 109]. As was shown in [48], the twisted backgrounds are unstable, and the instability is similar to that discussed in [118]. This instability is exponentially suppressed as $g_s \to 0$ and is likely to be completely absent when some supersymmetry is preserved. For the time being we will ignore the instability. We will return to this point in the discussion.

4.3.1 The T-dual of a Twist

We will first describe a type-II background without branes and then later we will add the brane probes. Consider type-IIA string theory on a space that is $\mathbb{R}^{9,1}$ modded out by the isometry

$$\mathcal{U} : (x_0, x_1, x_2, x_3, \{x_{3+a}\}_{a=1}^{6}) \mapsto (x_0, x_1, x_2, x_3 + 2\pi R_3, \{\sum_{b=1}^{6} O_{ba} x_{3+a}\}_{a=1}^{6}) \ . \tag{4.12}$$

Here $O \in SO(6)$ is an orthogonal matrix. The twisted compactification is parameterized by R_3 and, because we need to define the action on fermions, an element of $Spin(6) \cong SU(4)$. This background is, in general, modified by quantum corrections, but O and R_3 are defined by their asymptotic values at infinity. We will denote this background by $X(O, R_3)$. Note that if $R_3 > 0$, the isometry $\mathcal{U}$ has no fixed points and therefore O is not necessarily of finite order.

Now consider probing $X(O, R_3)$ with D2-branes in directions (x_0, x_1, x_2) and then taking the limit $R_3 \to 0$ together with $O = e^{\frac{2\pi i R_3 M}{\alpha'}}$, where M is a finite matrix of the Lie algebra $so(6) \cong su(4)$ with dimensions of length, and $\frac{\alpha'}{2\pi}$ is the inverse string tension.

When $M = 0$, we can perform T-duality to transform the D2-branes into D3-branes. When $M \neq 0$, we will now show the low energy description of the probe is a dipole theory.

4.3.2 Branes Probing Dual Twists

We wish to find the low energy Lagrangian describing D2-branes that probe the twisted geometry of subsection 4.3.1. The light degrees of freedom come from the strings with two Dirichlet boundary conditions, *i.e.*, fundamental strings with ends on the D2-branes. Because $R_3 \to 0$, we have to set the string oscillators to their ground states, but the winding number can be arbitrary.

To obtain the Lagrangian, we can adopt a procedure similar to the one described in [31, 108, 32] for noncommutative gauge theories. Also, the construction that we present here is reminiscent of the construction in [44]. In momentum space, the action of the dipole theory is obtained from the action of $\mathcal{N}=4$ SYM by inserting certain phases. Let $\Phi_1(p_1),\ldots,\Phi_n(p_n)$ be fields in the adjoint representation of $U(N)$ and suppose that $\mathcal{N}=4$ SYM has a term of the form

$$\mathrm{tr}\{\Phi_1(p_1)\cdots\Phi_n(p_n)\}\ , \tag{4.13}$$

in the Lagrangian (of course $n\leq 4$). The variables p_i are the momenta. Let the dipole vectors of the fields be $L_1,\ldots,L_n$. We have

$$\sum_{i=1}^{n} L_i = 0\ ,\qquad \sum_{i=1}^{n} p_i = 0\ . \tag{4.14}$$

The dipole theory is obtained from the ordinary $\mathcal{N}=4$ SYM theory by inserting the phases

$$e^{i\sum_{1\leq i<j\leq n} p_i L_j} \tag{4.15}$$

in front of terms like (4.13). Now let us consider branes probing $X(O,R_3)$. For simplicity, let us assume that the twist, O, acts only on $Z\equiv X^8+iX^9$ as $Z\to e^{i\alpha}Z$. We will refer to the angular momentum corresponding to rotation in the Z-plane as the *Z-charge*.

In the case that $\alpha=0$ we know that the theory on the D2-brane probe is $\mathcal{N}=4$ SYM. The states with momentum along the 3^{rd} direction, in the SYM theory, correspond to winding states along the 3^{rd} direction in the string theory setting.

Now let us turn on the twist, α. Consider a string disc amplitude that calculates the interaction of n open string states with winding numbers $w_1,\ldots,w_n$ and with *Z-charges* $q_1,\ldots,q_n$. The worldsheet theory has a global $U(1)$ symmetry corresponding to the Z charge. The string vertex operators that correspond to the external states

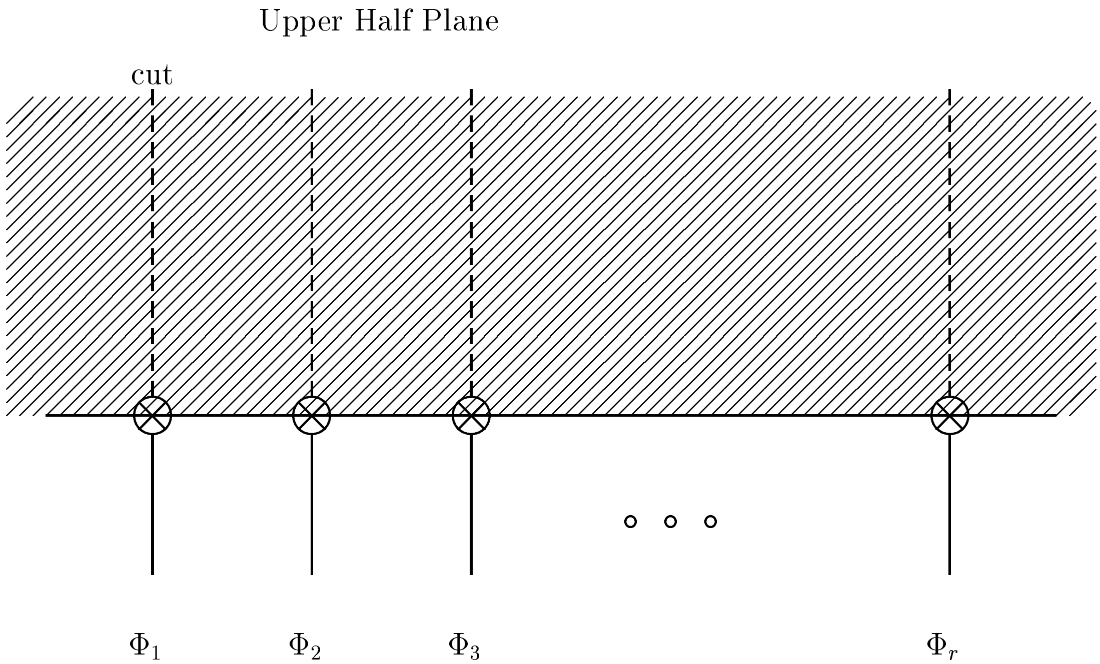


Figure 4.1: An r-point amplitude with vertex operators that carry winding numbers. It requires r cuts on the worldsheet.

are charged under this $U(1)$ symmetry. The disc worldsheet has cuts that emanate from the external vertex operators on the boundary. Along the j^{th} cut, the worldsheet field Z jumps by a phase $e^{i\alpha q_j}$. We can redefine the field Z to be continuous, but then there will be additional phases coming from the vertex operators on the boundary. It is easy to see that this phase is

$$e^{i\alpha \sum_{1\leq i<j\leq n} w_i q_j} \ . \tag{4.16}$$

This is illustrated in figure 4.1. It agrees with (4.15) because

$$L_j = \alpha' \frac{\alpha}{R_3} q_j \ , \qquad p_j = {\alpha'}^{-1} R_3 w_j \ . \tag{4.17}$$

In 3+1D, the photon of the $U(1) \subset U(N)$ center of the gauge group is likely to become massive via a dynamical mechanism similar to the one described in [46] for quiver theories, so that the gauge group is actually just $SU(N)$, but we will ignore this for the time being.

4.4 Supergravity Solution for a Twisted Brane

We now turn to the task of describing the supergravity duals of these dipole theories. We will use the string theory realization of dipole theories as described in the previous section.

We will find the exact *classical* supergravity solutions in four steps:

1. We start with the D3-brane solution of type-IIB classical supergravity and compactify one of the directions parallel to the D3-branes. We will call it the 3^{rd} direction.

2. We perform T-duality on the 3^{rd} direction to obtain a solution that describes D2-branes in type-IIA. The solution, however, will be translationally invariant along the 3^{rd} direction, and, as such, it describes smeared rather than localized D2-branes.

3. We now insert a transverse $SO(6)$ twist into the geometry by hand. This is accomplished by simply changing the boundary conditions for the 6 transverse coordinates as we complete a circle around the 3^{rd} direction. Locally, the metric is unchanged.

4. Finally, we use T-duality to turn the smeared D2-branes back to D3-branes.

We will restrict ourselves mostly to cases where all the dipole vectors of the fields in the theory are oriented in the same direction. This was direction 3 above. In section 4.8 we present the generalization for generic dipole vectors. We now turn to the details.

4.4.1 The Type-IIB D3-Brane

First, the conventions. We work in the $(+,-,\ldots,-)$ metric. Greek indices are $\mu,\nu = 0\ldots 2$. The time direction is $t = x_0$. The direction that we T-dualize is the 3^{rd}. The remaining directions, perpendicular to the brane, are labeled by roman indices $a,b = 4\ldots 9$. All metrics will be in string frame.

We start with the metric for a D3-brane (note that all the x's have dimensions of length)

$$ds^2_{\rm str} = H^{-\frac{1}{2}}(dt^2 - dx_1^2 - dx_2^2 - dx_3^2) - H^{\frac{1}{2}}(\delta_{ab}dx^a dx^b) \ , \tag{4.18}$$

where

$$H = 1 + \frac{\mathcal{R}^4}{r^4}, \qquad \mathcal{R}^4 = 4\pi g_s N \alpha'^2, \qquad r^2 = \delta_{ab}x^a x^b \ , \tag{4.19}$$

and we have the following backgrounds for the RR 4-from potential and the dilaton respectively

$$C^{(4)}_{0123} = -H^{-1}, \qquad e^{2\phi} = e^{2\phi_0} \ . \tag{4.20}$$

Next, we compactify along x^3 with radius $R_3 \equiv R$. The metric is now

$$ds^2 = H^{-\frac{1}{2}}(dt^2 - \delta_{\mu\nu}dx^\mu dx^\nu - R^2 d\hat{x}_3^2) - H^{\frac{1}{2}}(\delta_{ab}dx^a dx^b) \ . \tag{4.21}$$

Note that $\hat{x}_3$ is now dimensionless and periodic $\hat{x}_3 \sim \hat{x}_3 + 2\pi$.

4.4.2 A Smeared D2-Brane with a Twist

We now T-dualize around x_3. Following [24, 25, 19], we have

$$C^{(3)}_{012} = \tfrac{8}{3} H^{-1} \qquad e^{2(\phi-\phi_0)} = \tfrac{\alpha'}{R^2} H^{\frac{1}{2}}$$
$$ds^2 = H^{-\frac{1}{2}}\left(dt^2 - dx_1^2 - dx_2^2\right) - H^{\frac{1}{2}}\left(\tfrac{\alpha'^2}{R^2} d\hat{x}_3^2 + dx_4^2 + \cdots + dx_9^2\right) \quad . \tag{4.22}$$

This is a smeared D2-brane. We can now add a twist to the transverse directions $x_4, \cdots, x_9$ as we travel around the circle x_3. In particular, we take an element of the Lie algebra $so(6)$, Ω_{ab}, and change the metric to

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}(dt^2 - dx_1^2 - dx_2^2) \\ &- H^{\frac{1}{2}}\left\{\alpha'^2 R^{-2} d\hat{x}_3^2 + \sum_a \left(dx_a - \sum_b \Omega_{ab} x_b d\hat{x}_3\right)^2\right\} \quad . \end{aligned} \tag{4.23}$$

We can expand this out, giving

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}(dt^2 - dx_1^2 - dx_2^2) \\ &- H^{\frac{1}{2}}\left\{(\alpha'^2 R^{-2} + \vec{x}^\top \Omega^\top \Omega \vec{x}) d\hat{x}_3^2 + d\vec{x}^\top d\vec{x} - 2 d\vec{x}^\top \Omega \vec{x}\ d\hat{x}_3\right\} \ , \end{aligned} \tag{4.24}$$

where $\vec{x}$ is the vector formed by x_a $(a = 4 \ldots 9)$.

4.4.3 And Back to the D3-Brane

Once again, we apply the T-duality formulae (recall that $\hat{x}_3$ is dimensionless, while all the other coordinates have dimension of length)

$$
\begin{aligned}
ds^2 &= H^{-\frac{1}{2}}\left(dt^2 - dx_1^2 - dx_2^2 - \frac{\alpha'^2}{\alpha'^2 R^{-2} + \vec{x}^\top\Omega^\top\Omega\vec{x}} d\hat{x}_3^2\right) \\
&- H^{\frac{1}{2}}\left(d\vec{x}^\top d\vec{x} - \frac{(d\vec{x}^\top\Omega\vec{x})^2}{\alpha'^2 R^{-2} + \vec{x}^\top\Omega^\top\Omega\vec{x}}\right) . \qquad (4.25)
\end{aligned}
$$

We also have

$$
\begin{aligned}
C^{(4)}_{3012} &= H^{-1} \\
\sum_a B_{\hat{3}a} dx^a &= -\frac{d\vec{x}^\top\Omega\vec{x}}{\alpha'^2 R^{-2} + \vec{x}^\top\Omega^\top\Omega\vec{x}} \\
e^{2(\phi-\phi_0)} &= \frac{1}{1 + \alpha'^{-2} R^2 \vec{x}^\top\Omega^\top\Omega\vec{x}} , \qquad (4.26)
\end{aligned}
$$

which we will address later on.

Defining $\vec{x} = r\hat{n}$ so that $\|\hat{n}\| = 1$, our metric becomes

$$
\begin{aligned}
ds^2 &= H^{-\frac{1}{2}}\left(dt^2 - dx_1^2 - dx_2^2 - \frac{\alpha'^2}{\alpha'^2 R^{-2} + r^2\hat{n}^\top\Omega^\top\Omega\hat{n}} d\hat{x}_3^2\right) \\
&- H^{\frac{1}{2}}\left(dr^2 + r^2 d\hat{n}^\top d\hat{n} - \frac{r^4}{\alpha'^2 R^{-2} + r^2\hat{n}^\top\Omega^\top\Omega\hat{n}}(\hat{n}^\top\Omega^\top d\hat{n})^2\right) . \qquad (4.27)
\end{aligned}
$$

4.4.4 The Near-Horizon Limit

In these coordinates, the horizon is at $r = 0$, and so the near-horizon limit is r small. We can therefore approximate

$$
H^{\frac{1}{2}} = \sqrt{1 + \frac{\mathcal{R}^4}{r^4}} \sim \left(\frac{\mathcal{R}}{r}\right)^2 . \qquad (4.28)
$$

Substituting this into the metric, we obtain

$$
\begin{aligned}
ds^2 &= (r/\mathcal{R})^2 \left(dt^2 - dx_1^2 - dx_2^2 - \frac{\alpha'^2}{\alpha'^2 R^{-2} + r^2 \hat{n}^\top \Omega^\top \Omega \hat{n}} d\hat{x}_3^2 \right) \\
&- \frac{\mathcal{R}^2}{r^2} dr^2 - \mathcal{R}^2 \left(d\hat{n}^\top d\hat{n} - \frac{1}{\alpha'^2 (rR)^{-2} + \hat{n}^\top \Omega^\top \Omega \hat{n}} (\hat{n}^\top \Omega^\top d\hat{n})^2 \right) \ . \qquad (4.29)
\end{aligned}
$$

Finally, we make the substitution $u = \mathcal{R}^2/r$

$$
\begin{aligned}
ds^2 &= \frac{\mathcal{R}^2}{u^2} \left(dt^2 - dx_1^2 - dx_2^2 - \frac{u^2}{(\frac{u}{R})^2 + \lambda^2 \hat{n}^\top \Omega^\top \Omega \hat{n}} d\hat{x}_3^2 \right) - \mathcal{R}^2 \frac{du^2}{u^2} \\
&- \mathcal{R}^2 \left(d\hat{n}^\top d\hat{n} - \frac{\lambda^2}{(\frac{u}{R})^2 + \lambda^2 \hat{n}^\top \Omega^\top \Omega \hat{n}} (\hat{n}^\top \Omega^\top d\hat{n})^2 \right) \ , \qquad (4.30)
\end{aligned}
$$

where $\lambda^2 \equiv \frac{\mathcal{R}^4}{\alpha'^2} = 4\pi g_{\mathrm{YM}}^2 N$. The dilaton and the NSNS 2-form field are

$$
\begin{aligned}
\sum_a B_{\hat{3}a} d\hat{n}^a &= -\frac{\lambda^2}{\frac{u^2}{R^2} + \lambda^2 \hat{n}^\top \Omega^\top \Omega \hat{n}} d\hat{n}^\top \Omega \hat{n} \ , \\
e^{2(\phi - \phi_0)} &= \frac{1}{1 + \frac{R^2}{u^2} \lambda^2 \hat{n}^\top \Omega^\top \Omega \hat{n}} \ . \qquad (4.31)
\end{aligned}
$$

Now we take the limit $R \to \infty$ keeping $R\Omega = M$ fixed. We also redefine $\hat{x}_3 = \frac{x_3}{R}$. Note that x_3, u and M have dimensions of length. We find

$$
\begin{aligned}
ds^2 &= \frac{\mathcal{R}^2}{u^2} \left(dt^2 - dx_1^2 - dx_2^2 - du^2 - \frac{u^2}{u^2 + \lambda^2 \hat{n}^\top M^\top M \hat{n}} dx_3^2 \right) \\
&- \mathcal{R}^2 \left(d\hat{n}^\top d\hat{n} - \frac{\lambda^2}{u^2 + \lambda^2 \hat{n}^\top M^\top M \hat{n}} (\hat{n}^\top M^\top d\hat{n})^2 \right) \ . \qquad (4.32)
\end{aligned}
$$

The NSNS 2-form field and the dilaton are

$$
\begin{aligned}
\sum_a B_{3a} d\hat{n}^a &= -\frac{\lambda^2}{u^2 + \lambda^2 \hat{n}^\top M^\top M \hat{n}} d\hat{n}^\top M \hat{n} \ , \\
e^{2(\phi - \phi_0)} &= \frac{u^2}{u^2 + \lambda^2 \hat{n}^\top M^\top M \hat{n}} \ . \qquad (4.33)
\end{aligned}
$$

Given this form of the metric it is not obvious that the region $\frac{\mathcal{R}}{r} \gg 1$ indeed decouples from the bulk, as we have assumed. In principle, one can calculate scattering amplitudes for gravitons as in [65, 64]. In some cases one can see from the scattering amplitudes that the bulk does not decouple (see for instance [6]).

In our case, the geometry is strongly coupled when u is small, as will be discussed in more detail in subsection 4.6.1, and evaluating the scattering amplitudes is difficult. Nevertheless, there is no reason to expect that the bulk will not decouple. The dipole-theories describe well defined renormalizable theories that do not require additional degrees of freedom in the UV.

4.5 The Geometry of the Supergravity

We now investigate some of the geometrical features of the metric (4.32). The key things to note are the behavior of x_3 coordinate and the $\mathbf{S}^5$ as a function of u. We first discuss the generic behavior and then give a detailed analysis of a useful special case that will occupy us for the remainder of the chapter. General deformations of the $\mathbf{S}^5$ were also studied in a slightly different context in [50].

4.5.1 The Boundary

The behavior near the boundary is governed by the rank of M. For maximal rank, the quadratic function $\hat{n}^\top M^\top M \hat{n}$ is always positive definite. It has 12 local extrema on $\mathbf{S}^5$. These consist of pairs of antipodal points – each pair corresponds to an eigenvector of $M^\top M$ with the two ($\pm$) sign options. The metric (4.32) is asymptotically $\mathbf{AdS}_4 \times \mathbf{S}^1 \times \mathbf{S}^5$ where the $\mathbf{S}^5$ is deformed, and both the $\mathbf{S}^1$ and $\mathbf{S}^5$ are small compared to the **AdS**.

If the rank of M is less than maximal, the quadratic form $\hat{n}^\top M^\top M \hat{n}$ has a locus of zeroes. This locus is $\mathbf{S}^{r-1}$ where $r = 2, 4$ are the possible nonzero values for the rank. Locally on the zero locus, the metric is indistinguishable from ordinary $\mathbf{AdS}_5 \times \mathbf{S}^5$. This should be related to the fact that some scalar fields do not have a dipole length. We do not claim to understand the exact connection.

The metric on $\mathbf{S}^5$ becomes degenerate as $u \to 0$. For M of maximal rank, the metric on $\mathbf{S}^5$ at $u = 0$ is

$$ds^2 = \mathcal{R}^2 d\hat{n}^\top d\hat{n} - \frac{\mathcal{R}^2(\hat{n}^\top M^\top d\hat{n})^2}{\hat{n}^\top M^\top M\hat{n}} \ . \tag{4.34}$$

Let $\hat{n}$, a unit vector in $\mathbb{R}^6$, parameterize a point $p \in \mathbf{S}^5$. Then $M\hat{n}$ defines a direction in the tangent space $T_p\mathbf{S}^5$, since $\hat{n}^\top M\hat{n} = 0$. It is easy to see that the metric is degenerate along this direction. Thus, $M\hat{n}$ defines a vector field on $\mathbf{S}^5$ along which the metric is degenerate. This is the vector field induced by the infinitesimal $SO(6)$ action on $\mathbf{S}^5$ given by $M \in so(6)$. To analyze the degenerate $\mathbf{S}^5$ further we need to know more about the eigenvalues of M. Let the eigenvalues be $\pm i\alpha_1, \pm i\alpha_2, \pm i\alpha_3$. If $\alpha_1 = \alpha_2 = \alpha_3$ then the flow lines of the vector field $M\hat{n}$ are closed circles. $\mathbf{S}^5$ can be described as a circle bundle over $\mathbb{CP}^2$, and the vector field is along the circle. At $u = 0$ the $\mathbf{S}^5$ then shrinks to $\mathbb{CP}^2$. This particular case will be discussed more extensively in the next section. For the general case, we can identify $\mathbb{R}^6$ with $\mathbb{C}^3$ and introduce the following coordinates

$$(z_1, z_2, z_3) = \left(\frac{e^{i\alpha} r\cos\theta}{\sqrt{1+r^2}}, \frac{e^{i\beta} r\sin\theta}{\sqrt{1+r^2}}, \frac{e^{i\gamma}}{\sqrt{1+r^2}} \right) \ . \tag{4.35}$$

In these coordinates, the deformation of the sphere only affects the three coordinates α, β and γ. The vector $M\hat{n}$ is solely along this torus and, for generic ratios between these angles, the flow is dense in this torus. However, this is not a true fibration, and to avoid such complications we will only work with the simpler case.

4.5.2 The Hopf Fibration

The case when all three eigenvalues of M are equal is the case where all of the scalar fields have the same dipole length. The analysis of the UV behavior of the theory will significantly simplify in this situation.

We set $\frac{1}{2}\tilde{L} \equiv \alpha_1 = \alpha_2 = \alpha_3$ and

$$M = \begin{pmatrix} 0 & -\tilde{L} & 0 & 0 & 0 & 0 \\ \tilde{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tilde{L} & 0 & 0 \\ 0 & 0 & \tilde{L} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\tilde{L} \\ 0 & 0 & 0 & 0 & \tilde{L} & 0 \end{pmatrix} . \qquad (4.36)$$

For $\tilde{L} \neq 0$, this choice of M breaks all of the supersymmetry but it preserves a $U(3) \subset SO(6)$ subgroup of the R-symmetry. The advantage of this choice of M is that the factor $\hat{n}^\top M^\top M \hat{n} = \tilde{L}^2$ is independent of $\hat{n}$. According to the definition of M in subsection 4.2.2, the bosons have dipole lengths $\pm 2\pi\tilde{L}$, three of the fermions have dipole lengths $\pm\pi\tilde{L}$ and the remaining (complex) fermion has length $\pm 3\pi\tilde{L}$.

We now write the metric on the deformed $\mathbf{S}^5$ explicitly. Let a unit vector $\hat{n}$ which parameterizes $\mathbf{S}^5$ in $\mathbb{C}^3$ be given by

$$\hat{n} = \left(\frac{e^{i\gamma}}{\sqrt{1+|\alpha|^2+|\beta|^2}}, \frac{e^{i\gamma}\alpha}{\sqrt{1+|\alpha|^2+|\beta|^2}}, \frac{e^{i\gamma}\beta}{\sqrt{1+|\alpha|^2+|\beta|^2}} \right) . \qquad (4.37)$$

with α and β complex. Thus, the $\mathbf{S}^5$ is given as a circle fibration parametrized by γ over $\mathbb{CP}^2$ parametrized by α and β. This is the famed Hopf fibration. The advantage of these coordinates is that the vector $M\hat{n}$ points along the direction of the fiber for

M given as in (4.36).

It can be shown that the metric on a regular $\mathbf{S}^5$ is in these coordinates

$$\begin{aligned} d\hat{n}^\top d\hat{n} &= \frac{|d\alpha|^2 + |d\beta|^2}{1 + |\alpha|^2 + |\beta|^2} - \frac{|\bar{\alpha} d\alpha + \bar{\beta} d\beta|^2}{(1 + |\alpha|^2 + |\beta|^2)^2} \\ &+ \left(d\gamma + \frac{\mathrm{Im}\,(\bar{\alpha} d\alpha + \bar{\beta} d\beta)}{1 + |\alpha|^2 + |\beta|^2} \right)^2 , \end{aligned} \tag{4.38}$$

where the first 2 terms describe the Fubini-Study metric on $\mathbb{CP}^2$.

For our deformed sphere, the metric is

$$\begin{gathered} d\hat{n}^\top d\hat{n} - \frac{\lambda^2 (\hat{n}^\top M^\top d\hat{n})^2}{u^2 + \lambda^2 \hat{n}^\top M^\top M \hat{n}} = \\ \frac{|d\alpha|^2 + |d\beta|^2}{1 + |\alpha|^2 + |\beta|^2} - \frac{|\bar{\alpha} d\alpha + \bar{\beta} d\beta|^2}{(1 + |\alpha|^2 + |\beta|^2)^2} + \\ \frac{u^2}{u^2 + \lambda^2 \tilde{L}^2} \left(d\gamma + \frac{\mathrm{Im}\,(\bar{\alpha} d\alpha + \bar{\beta} d\beta)}{1 + |\alpha|^2 + |\beta|^2} \right)^2 . \end{gathered} \tag{4.39}$$

The 5×5 determinant of the above metric can be calculated to be

$$\det\, g = \left(\frac{u^2}{u^2 + \lambda^2 \tilde{L}^2} \right) \frac{1}{(1 + |\alpha|^2 + |\beta|^2)^6} \tag{4.40}$$

Thus, the salient features of our deformed $\mathbf{S}^5$ are as follows

- It has the structure of an $\mathbf{S}^1$ (Hopf) fibration over a base $\mathbb{CP}^2$. An $SU(3)$ subgroup of $SO(6)$ acts freely on $\mathbb{CP}^2$.

- Invariance of the metric of the deformed $\mathbf{S}^5$ under $U(3) \subset SO(6)$ implies that the metric on the base $\mathbb{CP}^2$ is independent of the position, and the metric on the fiber $\mathbf{S}^1$ is similarly homogeneous due to the $U(1)$ isometry which rotates the fibers.

- The radius of the fiber is independent of the $\mathbb{CP}^2$ coordinate and is given by

$$\rho(u) = \mathcal{R}\frac{u}{\sqrt{u^2+\lambda^2\tilde{L}^2}} \ . \tag{4.41}$$

- The volume of $\mathbb{CP}^2$ is constant and given by

$$\text{Volume}(\mathbb{CP}^2) = \frac{\pi^2}{2}\mathcal{R}^4 \ . \tag{4.42}$$

Finally, in these coordinates, the NSNS 2-form is given by

$$B = -\frac{\lambda^2\tilde{L}}{u^2+\lambda^2\tilde{L}^2}dx_3\wedge\psi \ . \tag{4.43}$$

Here ψ is the global angular 1-form of the Hopf fibration. In the notation of (4.38), it is given by

$$\psi = d\gamma + \frac{\text{Im}\left(\bar{\alpha}d\alpha+\bar{\beta}d\beta\right)}{1+|\alpha|^2+|\beta|^2} \ . \tag{4.44}$$

The 3-form field strength is given by

$$H = dB = -\frac{\lambda^2\tilde{L}}{u^2+\lambda^2\tilde{L}^2}dx_3\wedge d\psi + \frac{\lambda^2\tilde{L}u}{(u^2+\lambda^2\tilde{L}^2)^2}du\wedge dx_3\wedge\psi \ . \tag{4.45}$$

Here $d\psi$ is the closed harmonic 2-form that generates $H^2(\mathbb{CP}^2,\mathbb{Z})$.

4.6 Nonlocality in the Supergravity Dual

In this section, we will show how the nonlocality of the field theory is manifested in the geometry of the boundary of the supergravity. We will continue to work with the special case described above in 4.5.2. In this situation, as described in that section, the fiber shrinks to zero size on the boundary, and, as such, should be T-dualized to obtain a classical description. This will make the dipole nature of the nonlocality evident.

4.6.1 T-duality of the Fiber

As we approach the boundary of our solution, $u \to 0$, the volume of the base $\mathbb{CP}^2$ remains a constant. However, the circle fibered along it shrinks to zero size. Note that the dilaton also approaches zero since

$$e^{2(\phi-\phi_0)} = \frac{u^2}{u^2+\lambda^2\tilde{L}^2} \ . \tag{4.46}$$

It is easy to see that the curvature of the deformed $\mathbf{S}^5$ is still of the order of magnitude of $\frac{1}{\mathcal{R}^2}$, even when $u \ll \lambda\tilde{L}$. However, when $\rho(u)$ becomes of the order of magnitude of the string length, $\alpha'^{1/2}$, we cannot trust the supergravity approximation anymore. This happens when $u \sim \alpha'^{1/2}\mathcal{R}^{-1}\lambda\tilde{L} = \lambda^{1/2}\tilde{L}$.

Since the circle shrinks to zero, we have to perform T-duality on that direction. There is a subtlety that complicates matters when we consider fermions [16], but we will naively apply the standard T-duality formulae anyway.

Again using the equations of [24, 25, 19], we obtain type-IIA with the metric

$$\begin{aligned} ds^2 &= \frac{\mathcal{R}^2}{u^2}(dt^2 - dx_1^2 - dx_2^2 - du^2) - \frac{\mathcal{R}^2}{u^2}(dx_3 + \tilde{L}d\gamma)^2 - \frac{\alpha'^2}{\mathcal{R}^2}d\gamma^2 \\ &\quad -(\text{constant } \mathbb{CP}^2) \ . \end{aligned} \tag{4.47}$$

We also have

$$\begin{aligned} e^{2\phi} &= \frac{e^{2\phi_0}}{\lambda} = \sqrt{\frac{g_s^3}{4\pi N}}, \\ \sum_b H_{u3b}dx^b &= -\frac{\lambda^2\tilde{L}u}{(u^2+\lambda^2\tilde{L}^2)^2}\frac{\operatorname{Im}(\bar{\alpha}d\alpha + \bar{\beta}d\beta)}{1+|\alpha|^2+|\beta|^2} \ . \end{aligned} \tag{4.48}$$

where H is the 3-form NSNS field strength. In addition, there is a non-trivial RR 4-form field strength which we will not write down. Note that the type-IIA dilaton becomes a constant. Despite the ominous factor $\frac{\alpha'^2}{\mathcal{R}^2} \ll 1$ in (4.47), we see that type-IIA supergravity is a good approximation. No two points that are closer than $\alpha'^{1/2}$ are identified. The only identification is

$$(\ldots, x_3, \gamma) \sim (\ldots, x_3, \gamma + 2\pi) \tag{4.49}$$

and the distance between those two points is large when $u \to 0$.

4.6.2 Nonlocality on the Boundary

The metric in equation (4.47) is a striking manifestation of the nonlocality of the field theory in the boundary metric. It describes the x_3 direction fibered over a small circle of radius $\frac{\alpha'}{\mathcal{R}}$ parameterized by γ. The proper distance between the point with coordinates (x_3, γ) and the point with coordinates $(x_3 + 2\pi\tilde{L}, \gamma) \sim (x_3, \gamma - 2\pi)$ is $\frac{2\pi\alpha'}{\mathcal{R}}$ which is of stringy scale. On the other hand, the proper distance between (x_3, γ) and $(x_3 + \Delta, \gamma)$ is of order $\frac{\mathcal{R}}{u} \to \infty$ when Δ is not an integer multiple of $2\pi\tilde{L}$ and $u \to 0$. In the field theory, this is translated into nonlocal interactions between fields at points that are separated by a distance of $L = 2\pi\tilde{L}$. If we think of the matter content of the dual SYM theory as constituting momentum modes along the $\mathbf{S}^5$, then, after T-duality, the nonlocality should be reflected in the winding number around the T-dual circle. This is exactly what we see here.

The metric (4.47) also shows that the 4D superconformal group is restored since the new coordinate $x_3 + \tilde{L}\gamma$ can be attached to the $\mathbf{AdS}_4$ part of the metric to form $\mathbf{AdS}_5$. This is to be expected because the nonlocal interactions have a minimal distance L. At short distances the vicinity of each point should look like a 4D CFT and the interactions with fields at distance L seems like an interaction with extra degrees of freedom outside the small neighborhood of the point.

4.6.3 A Note on Momentum Conservation

It is interesting to note that, because $\mathbf{S}^5$ is contractible, the winding number along the $\mathbf{S}^1$ fiber is not conserved. This is equivalent to the fact that the fibration has a nontrivial first Chern class. In order to contract the circle, however, one needs to pull it around a nontrivial 2-cycle of the base $\mathbb{CP}^2$. So, a concrete process that violates winding number conservation is to start with a small string on $\mathbb{CP}^2$ and then to gradually increase its size until it extends around the equator of a topologically nontrivial $\mathbb{CP}^1 \cong \mathbf{S}^2$ inside $\mathbb{CP}^2$. Then we contract the string along the other hemisphere of the $\mathbb{CP}^1$. At the end of the process, the string is wound around the fiber $\mathbf{S}^1$. This process requires energy scales of the order of the circumference of the equator of the $\mathbb{CP}^1$, *i.e.*, $E \sim \mathcal{R}/\alpha'$

Because of this, after T-duality, momentum along the γ-direction also must not be conserved. After T-duality, the γ-circle is fibered trivially over the $\mathbb{CP}^2$. Instead, we have a 3-form NSNS field strength, $H_{\gamma ab}$, along the circle and two directions inside the $\mathbb{CP}^2$. It is easy to see that $H_{\gamma ab}$ is proportional to $d\gamma \wedge \omega$ where ω is the harmonic 2-form on $\mathbb{CP}^2$.

The process that violates momentum conservation along the γ-direction is the same as before. We start with a pointlike string inside $\mathbb{CP}^2$ and deform it to go around a nontrivial 2-cycle inside $\mathbb{CP}^2$ and then shrink it back to a point. Let $X(\sigma, \tau)$ be the closed path of the string as a function of time τ and string coordinate $0 \leq \sigma \leq 2\pi$. Note that when both σ and τ vary, the function $X(\sigma, \tau)$ spans a surface that is homologically equivalent to the nontrivial 2-cycle inside $\mathbb{CP}^2$. The violation of momentum conservation is due to the "magnetic" forces on a moving string in the presence of an $H = dB$ field strength. The total γ-momentum transfer is

$$\int F_\gamma(\tau)d\tau = \int H_{\gamma ab}\partial_\sigma X^a \partial_\tau X^b d\sigma d\tau = \int \omega = 1\ . \tag{4.50}$$

The RHS is the integral of the 2-form ω along the nontrivial 2-cycle.

4.7 Correlation Functions

In a local field theory, correlation functions of operators, $\langle O(x)O(y)\rangle$, have short distance singularities when $x \to y$. In dipole theories, we expect a singularity to appear also when $x \to y \pm L_i$, where L_i is one of the characteristic vectors of nonlocality as in section 4.2. In the special case we study in this chapter, the length of the characteristic vectors of the scalars is $L = 2\pi\tilde{L}$. For operators $O(x)$ that have no dipole length of their own (for example $\mathrm{tr}\{F_{\mu\nu}^2\}$) we therefore expect

$$\langle O(x)O(y)\rangle \longrightarrow_{x\to y+L} \frac{C}{|x-y-L|^{2\Delta}} \tag{4.51}$$

and then in momentum space we expect to find a term that behaves like

$$\langle O(k)^\dagger O(k)\rangle \longrightarrow_{k\to\infty} \frac{Ce^{ik\cdot L}}{k^{4-2\Delta}}\ . \tag{4.52}$$

For operators $O(x)$ that do have a length we expect the behavior of the correlation function to be more complicated since the operators contain nonlocal Wilson lines. It is likely that the correlation functions exhibit an exponential behavior $\sim e^{\sqrt{(\mathrm{const})|k_3|L}}$ analogous to that of noncommutative geometry [59, 98]. [1] For the rest of this discussion we will restrict ourselves to operators $O(x)$ with dipole length zero.

[1] We are grateful to M. Rozali for a discussion on this point.

We can use the AdS/CFT correspondence to compute these correlation functions in the large N limit. We will restrict ourselves to the special case where the R-symmetry is broken from $Spin(6)$ down to $U(3)$, as in section 4.5.2. Because the AdS/CFT correspondence directly probes the nonperturbative nature of the field theory, it is perhaps a bit too much to expect to see the exact form above, but, in the limit of high momentum along the dipole direction, a sign of nonlocality would be a rapid oscillation in the correlation function in momentum space.

It is, in general, a difficult problem to decouple the fields on a nontrivial background such as any of the examples in this chapter. Following [59], we will simply postulate that there exists a massless scalar living on our spacetime.[2] In particular, it should satisfy the field equation

$$\partial_\mu \left(e^{-2\phi} \sqrt{\det\, g}\; g^{\mu\nu} \partial_\nu \Phi(\vec{x}, u) \right) = 0 \ , \qquad (4.53)$$

where $\vec{x} = (t, x_1, x_2, x_3)$.

This is still quite a difficult problem to solve, but we will soon see how it can be simplified. In particular, we recall the determinant of the metric of the sphere, (4.40), in the Hopf fibration coordinates. Including the **AdS** portion of the metric, we have

$$\det\, g = \mathcal{R}^{20} \left(\frac{u^2}{u^2 + \lambda^2 \tilde{L}^2} \right)^2 u^{-10} \frac{1}{(1 + |\alpha|^2 + |\beta|^2)^6} \ . \qquad (4.54)$$

We immediately see that this factors into a contribution that depends on the sphere and one that depends on the **AdS**. Thus, because our metric is block diagonal, we can choose our scalar field to be constant on the sphere, and all contributions from the sphere will cancel out of our equations. Another happy fact is that the contribution from the dilaton exactly cancels the $u^2/(u^2 + \lambda^2 \tilde{L}^2)$ term reducing this to almost the

[2] We are grateful to I.R. Klebanov for explaining the relevant issues to us.

standard massless field equation on **AdS** space.

As usual, the most interesting part of the equation comes from the u coordinate, so we write

$$\Phi(\vec{x}, u) = \varphi(u) e^{i\vec{k}\cdot\vec{x}} \ . \tag{4.55}$$

Then φ satisfies the following equation

$$u^3 \partial_u \left(\frac{1}{u^3} \partial_u \varphi(u) \right) + \left(k^2 - \frac{(\lambda k_3 \tilde{L})^2}{u^2} \right) \varphi(u) = 0 \ . \tag{4.56}$$

If we expand this, we obtain

$$\varphi'' - \frac{3}{u}\varphi' + \left(k^2 - \frac{(\lambda k_3 \tilde{L})^2}{u^2} \right) \varphi = 0 \ . \tag{4.57}$$

We recognize this as the equation for a massive field in ordinary **AdS** space with $m\mathcal{R} = \lambda k_3 \tilde{L}$. Thus, we can copy the final result from equation (44) of [63]

$$\langle \mathcal{O}(k)\mathcal{O}(q) \rangle = -(2\pi)^4 \delta^4(k+q) \frac{N^2}{8\pi^2} \frac{\Gamma(1-\nu)}{\Gamma(\nu)} \left(\frac{k\mathcal{R}}{2} \right)^{2\nu} \mathcal{R}^{-4} \ , \tag{4.58}$$

where $\nu = \sqrt{4 + (k_3 \tilde{L})^2}$.

Let us now take the limit that $k_3 \to \infty$. In this limit, we have

$$\langle \mathcal{O}(k)\mathcal{O}(-k) \rangle \sim \frac{1}{\sin(\pi\nu)} \left(\frac{(k\mathcal{R}/2)^\nu}{\Gamma(\nu)} \right)^2 \underset{k_3 \to \infty}{\sim} \frac{\lambda k_3 \tilde{L} \left(\frac{|k| e \mathcal{R}}{2\lambda k_3 \tilde{L}} \right)^{2\lambda k_3 \tilde{L}}}{\sin(\pi \lambda k_3 \tilde{L})} \ . \tag{4.59}$$

It exhibits an oscillatory behavior but not quite what we have anticipated. We expected the wavelength of the nonlocal behavior to be an integer multiple of the dipole length. This is not what we observe here. This is a puzzling phenomenon, but it is consistent with the observation from the supergravity dual that the scale of the nonlocality is actually $\lambda\tilde{L}$ rather than just $\tilde{L}$. Since $\lambda\tilde{L} >> \tilde{L}$ there is no immediate contradiction. It could be that in the large λ limit the dominant contribution to the nonlocal behavior of the correlation function comes from the nonlocality on scale $[\lambda]\tilde{L}$ (where $[\lambda]$ is the integer that is closest to λ). It is important to realize, however, that the supergravity approximation ceases to be valid when $u < \alpha'^{1/2}\mathcal{R}^{-1}$, as we explained in subsection 4.6.1. This suggests that the above calculation may not be entirely valid. This is worthy of further investigation.

4.8 Generic Orientation of the Dipole Vectors

In section 4.4 we promised to describe the supergravity solution for generic dipole theories where the various dipole vectors are not all along the same direction. In order to avoid clutter, we will set $\alpha' = 1$ in this section.

We start with a D3-brane extended in the 0123 directions, compactified on a T^3 with radii (R_1, R_2, R_3). The relevant non-zero fields are

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H}}\left(dt^2 - (R_1 dx^1)^2 - (R_2 dx^2)^2 - (R_3 dx^3)^2\right) - \sqrt{H}(dx^a)^2 \\ C_{0123} &= -\frac{1}{H} \\ \varphi &= \varphi_0 \ , \end{aligned} \tag{4.60}$$

where

$$H = 1 + \frac{\mathcal{R}^4}{r^4} \qquad\qquad r^2 \equiv (x^a)^2 \ . \tag{4.61}$$

The roman indices $a, b, \ldots$ run from 4 to 9, and we use greek indices to indicate the compactified directions 1,2,3. Starting from the solution (4.60), we perform the T-duality transformation three times, in the three compactified directions using the formulae of [24, 25, 19]. The answer, which is a D0-brane smeared over the T-dual torus $T^3 : (R_1^{-1}, R_2^{-1}, R_3^{-1})$, is

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H}}dt^2 - \sqrt{H}\left(\left(\frac{dx^1}{R_1}\right)^2 + \left(\frac{dx^2}{R_2}\right)^2 + \left(\frac{dx^3}{R_3}\right)^2\right) - \sqrt{H}(dx^a)^2 \\ C_0^{(1)} &= -\frac{4}{H} \\ e^{2(\phi-\varphi_0)} &= \frac{H^{3/2}}{R_1^2R_2^2R_3^2} \ . \end{aligned} \tag{4.62}$$

Now, we introduce the three twists, by replacing

$$dx^a \longrightarrow dx^a - \sum_\mu (\Omega^\mu_{ab}x^b)dx^\mu \ , \tag{4.63}$$

where $(\Omega^\mu)^\top = -\Omega^\mu$ are commuting elements of SO(6). The metric with the twist is

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H}}dt^2 - \sqrt{H}\left(\left(\frac{dx^1}{R_1}\right)^2 + \left(\frac{dx^2}{R_2}\right)^2 + \left(\frac{dx^3}{R_3}\right)^2\right) \\ &- \sqrt{H}\left(dx^a - (\Omega^\mu_{ab}x^b)dx^\mu\right)^2 \ . \end{aligned} \tag{4.64}$$

We T-dualize three times to get back the metric for a D3-brane with a dipole theory living on it. Define $M^\mu \equiv R_\mu \Omega^\mu$ (no contraction over μ) and $x^a \equiv r\hat{n}^a$ where $\hat{n}^\top \hat{n} = 1$. With some work, the metric turns out to be (here and below there is no contraction in terms like $R_\nu dx^\nu$)

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H}}dt^2 - \sqrt{H}dr^2 \qquad (4.65) \\ &- \frac{1}{\sqrt{H}}\frac{\epsilon_{\alpha\beta\gamma}\epsilon_{\kappa\mu\nu}\left[\delta^{\alpha\kappa} + r^2(m^\alpha)^\top m^\kappa\right]\left[\delta^{\beta\mu} + r^2(m^\beta)^\top m^\mu\right]}{2D}(R_\gamma dx^\gamma)(R_\nu dx^\nu) \\ &- \sqrt{H}(r^2 dn^T dn) \\ &+ \sqrt{H}\left(\frac{r^4\epsilon_{\alpha\beta\gamma}\epsilon_{\kappa\mu\nu}\left[\delta^{\alpha\kappa} + r^2(m^\alpha)^\top m^\kappa\right]\left[\delta^{\beta\mu} + r^2(m^\beta)^\top m^\mu\right]\left[((m^\gamma)^\top dn)((m^\nu)^\top dn)\right]}{2D}\right) \ . \end{aligned}$$

where we have defined

$$D \equiv \frac{1}{6}\epsilon_{\alpha\beta\gamma}\epsilon_{\kappa\mu\nu}\left[\delta^{\alpha\kappa} + r^2(m^\alpha)^\top m^\kappa\right]\left[\delta^{\beta\mu} + r^2(m^\beta)^\top m^\mu\right]\left[\delta^{\gamma\nu} + r^2(m^\gamma)^\top m^\nu\right] \quad (4.66)$$

and

$$m^\alpha \equiv M^\alpha \hat{n} \; . \quad (4.67)$$

The other nonzero fields are

$$\begin{aligned} C^{(4)}_{0123} &= \frac{1}{H} \\ B^{(1)}_{\mu a} dx^\mu \wedge d\hat{n}^a &= -\sqrt{H} j_{\gamma\nu} dx^\gamma \wedge \frac{r m^\mu d\hat{n}}{R_\nu} \\ &= \frac{r\epsilon_{\alpha\beta\gamma}\epsilon_{\kappa\mu\nu}\left[\delta^{\alpha\kappa} + r^2(m^\alpha)^\top m^\kappa\right]\left[\delta^{\beta\mu} + r^2(m^\beta)^\top m^\mu\right]\left[(R_\gamma dx^\gamma)\wedge((m^\nu)^\top d\hat{n})\right]}{2D} \\ e^{2(\varphi-\varphi_0)} &= \frac{1}{D} \; . \end{aligned} \quad (4.68)$$

For $M^1 = M^2 = 0$ this reduces to the answers for a single twist. It is interesting to ask what happens when the twists, M^μ, do not commute. In this situation, the Ricci scalar of the twisted metric (4.64) has a field strength term, and thus the metric is no longer a solution to the supergravity equations.

4.9 Discussion

We have shown how the nonlocality of dipole theories is manifested in the supergravity dual. We discovered that the metric becomes degenerate at the boundary of the spacetime and that this could be used to explicitly demonstrate the nonlocality. Although this feature of the metric was shown using the naive T-duality to type-IIA and, as was argued in [16], one actually gets type-0A with a strong RR field strength, we believe that the metric still has this general structure. This should be a generic feature of the supergravity duals of nonlocal field theories. It is not a surprising result. Nonlocality, when realized in some limit of string theory, cannot be a purely supergravity effect. The nonlocality must be a result of the inclusion of some stringy degrees of freedom on the boundary. The degeneracy of the metric in string frame means that we cannot treat the boundary as classical, and this is the source of the nonlocality.

It is worthwhile to compare this situation to that in noncommutative geometry to see if we can distill some more general features of the supergravity dual. The discussion that follows has some features in common with [44],[73, 76, 8, 100]. [3]

Recall that the metric of the supergravity dual of NCSYM is [72, 84] (ignoring dimensionless constants)

$$ds^2 = \frac{1}{u^2}\left(dt^2 - dx_1^2 - \frac{u^4}{u^4+\theta^2}\left(dx_2^2 + dx_3^2\right) - du^2\right) \,. \tag{4.69}$$

The other fields are

$$\begin{aligned} e^{2\phi} &= \frac{u^4}{u^4+\theta^2} \\ B_{23} &= -\frac{\theta}{u^4+\theta^2} \,. \end{aligned}$$

We see that both the second and third directions go to zero length on the boundary, indicating some sort of stringy effect. Note that here, the degeneracy is in the **AdS** part of the metric indicating that the nonlocality is part of the space that the field theory lives on. This is in contrast to our dipole theories where the degeneracy is on the $\mathbf{S}^5$ indicating that the nonlocality is part of the field content of the theory.

Following the same procedure as in 4.6.2, we compactify these directions and T-

[3]We are grateful to A. Hashimoto for pointing out some of these references and for discussing this with us.

dualize along one of them, say the second. As before, the presence of the B-field gives rise to cross terms in the metric. Specifically, after T-duality, we have, isolating the 2 and 3 directions

$$ds^2 = (u^4 + \theta^2)\frac{dx_2^2}{u^2} + 2\theta\frac{dx_2 dx_3}{u^2} + \frac{u^2 + \theta^2 u^{-2}}{u^4 + \theta^2}dx_3^2 \ . \qquad (4.70)$$

If we take the $u \to 0$ limit, we can rewrite this as

$$ds^2 = \frac{1}{u^2}(\theta dx_2 + dx_3)^2 \qquad (4.71)$$

This has almost the same form as the metric we obtained in section 4.6.1. When we traverse the 2-circle, the above coordinate gets shifted by θ. As T-duality interchanges momentum with winding, we interpret this as a dipole in the 3 direction with length equal to θ times the momentum. This is exactly the situation in NCSYM.

What are the general features of the supergravity duals of nonlocal field theories that we can infer from this?

- The metric becomes degenerate on the boundary of **AdS**.
- The NSNS 2-form field has a component along the degenerate direction.
- We can (perhaps after compactification) T-dualize along this direction.
- After T-duality, the NSNS 2-form field induces off-diagonal terms in the metric that can be interpreted as a fibration over a string scale circle.
- The nonlocality of the field theory is manifested by the shift in the new coordinate as we go around the string scale circle.

While these features may not be generic for all nonlocal theories, it is not unreasonable to assume that they may be generic for the generalized dipole theories mentioned at the end of section 4.2.1 of which both the dipoles discussed here and those of noncommutative geometry are a special case. In [17] a generalization of dipoles to the case of the (2,0) theory was proposed where, instead of constant length dipoles, there are constant area "discpoles". This should have a supergravity dual of the from $\mathbf{AdS}_7 \times \mathbf{S}^4$. It would be interesting to investigate the effects of nonlocality on the supergravity in this situation.

Before concluding let us return to a loose end from the beginning of section 4.3. We mentioned that the twisted string theory backgrounds are unstable if supersymmetry is broken. This instability was discussed in [48, 36, 109] and is related to the instability of Kaluza-Klein compactifications without supersymmetry [118]. In section 4.5.2 we used a nonsupersymmetric twisted theory, and we therefore expect it to be unstable. However, the probability for decay per unit time and volume is exponentially suppressed as $g_s \to 0$. In the large N limit (keeping $g_s N$ fixed) we can therefore assume that the background is stable. It is interesting to ask whether the dipole field theory on the probe is also unstable. We will not address this question here. One possibility suggested by O. Aharony is that a potential is generated on the Coulomb branch of the dipole field theory that makes the origin unstable. This is currently under investigation.

Chapter 5

M(atrix)-Theory Scattering on the Noncommutative Five-brane

In this chapter, we study the quantum mechanics on the hyper-Kähler manifold that is the blow-up of an A_1-singularity. This system is relevant for M(atrix)-theory as it was conjectured to describe scattering in the "noncommutative" deformation of the superconformal (2,0) theory living on the worldvolume of a stack of k five-branes, which is a free 5+1D tensor multiplet. We study the M(atrix)-model in the sector with two units of longitudinal light-like momentum.

5.1 Introduction

As was mentioned in section 1.3, the five-brane of M-theory is not a very well understood object.

A single five-brane is described at low energies by a free tensor multiplet with $\mathcal{N} = (2,0)$ supersymmetry. An extension of the free tensor multiplet to an interacting theory has been suggested in [4] and [89, 20]. It was motivated by the string theory realization [34, 44] of gauge theories on a noncommutative $\mathbb{R}^4$. This conjectured 5+1D theory breaks Lorenz invariance explicitly. It is assumed to depend on a constant anti-self-dual 3-form parameter Θ^{ijk} of dimension $(\text{Mass})^{-3}$. At low energies the theory reduces to the free tensor multiplet.

In [104] a consistent limit of the five-brane with a strong 3-form field-strength was presented and it was suggested that at low-energies this limit describes a decoupled noncommutative $(2,0)$-theory. This is similar to the noncommutative theory obtained on a D-brane in the presence of a magnetic field. For a single D4-brane, the action of the worldvolume $U(1)$ gauge-theory on a noncommutative $\mathbb{R}^4$ can be expanded as the free action plus terms that are of higher order (in the noncommutativity) and depend only on the magnetic field field and its derivatives [104]. Similarly, the equations of motion of the noncommutative (5+1)D theory can, presumably, be expanded in Θ^{ijk}. We will derive the leading terms in section 5.2.

For a stack of five-branes, the low energy theory which lives on the world-volume of these branes is a superconformal U(k) gauge theory with $(2,0)$ supersymmetries. The theory contains k tensor multiplets, each consisting of five scalars, a self-dual field strength (with 3 independent components) and a (eight-dimensional) fermion, see [3] and references within.

One of the exciting features about the interacting noncommutative theory is that it has a conjectured M(atrix)-model [15] that is described in terms of a quantum mechanics on a nonsingular space [4, 89, 20].

Recall that M-theory, when compactified on a small circle, becomes type IIA string theory. Considering this dual type IIA theory helps clarify the problem of the five-branes. When we compactify, on a direction parallel to the five-branes, the duality between M-theory and type IIA theory turns five-branes into D4-branes. These D4-branes are described by a U(k) gauge theory. Boosting the original M-theory along the compact direction by N units of compact momentum produces N D0-branes. It is known that D0-branes can bind to D4-branes; in fact, N D0-branes bound to k D4-branes can be thought of as N instantons in the U(k) gauge theory living on the k D4-branes [43]. The correct Matrix model for the $(2,0)_k$ theory is thus SUSY QM, and the target space on which it 'lives' must be the space $\mathcal{M}_{N,k}$ of N instantons of U(k) theory on R^4 [3, 4]. This space can be obtained by the usual ADHM [4, 9, 89] construction, and is given by [89, 20]

$$\begin{pmatrix} [X,X^\dagger]+[Y,Y^\dagger]+Z^\dagger Z - W^\dagger W & = & 0 \\ [X,Y]+Z^\dagger W & = & 0 \end{pmatrix} /U(N) \ , \tag{5.1}$$

where X,Y are complex $N \times N$ matrices; Z,W are complex $k \times N$ matrices and the action of U(N) which we must quotient out by hand is

$$X \to gXg^{-1} \ , \qquad Y \to gYg^{-1} \ , \qquad Z \to Zg^\dagger \ , \qquad W \to Wg^\star \ . \tag{5.2}$$

Due to the $U(N)$ quotient, the manifold $\mathcal{M}_{N,k}$ is singular at the origin. This singularity, arising from conformal invariance (implying that any instanton can be rescaled to arbitrarily small size) needs to be resolved. The proposed resolution [89, 20] is to introduce a finite nonlocality into the theory, via noncommutative geometry.

Following [89], modify $\mathcal{M}_{N,k}$ to define a smooth manifold $\tilde{\mathcal{M}}_{N,k}$ by

$$\begin{pmatrix} [X,X^\dagger]+[Y,Y^\dagger]+Z^\dagger Z - W^\dagger W & = & \xi^2 Id_{N\times N} \\ [X,Y]+Z^\dagger W & = & 0 \end{pmatrix} /U(N) \ . \tag{5.3}$$

The conical singularity at the origin has been blown up to a sphere of radius ξ. This modified moduli space works in the ADHM construction when we replace R^4 with its noncommutative version. Defining $z_1 = x^1 + ix^2$ and $z_2 = x^3 + ix^4$, we postulate that these coordinates do not commute:

$$[z_1, \bar{z}_1] = [z_2, \bar{z}_2] = -\frac{\xi^2}{2} \ . \tag{5.4}$$

The ADHM construction now follows. Thus, $\tilde{\mathcal{M}}_{N,k}$ parameterizes the space of instantons, but in the U(k) gauge theory on noncommutative R^4. This is an interesting example of using a noncommutative deformation of a theory act as a regulator, in this case introducing a scale into the theory.

The simplest nontrivial sector of this M(atrix)-model is the sector with longitudinal light-cone momentum $p_{||} = 2/R_{||}$, where $R_{||}$ is the radius of the light-like direction. It describes the scattering of two massless particles, corresponding to the tensor multiplet, each with longitudinal, light-cone, momentum of $p_{||} = 1/R_{||}$. This is similar to the calculations of scattering of gravitons and their supersymmetric partners [45, 90, 112] in the M(atrix)-model of 10+1D M-theory. The M(atrix)-model for the noncommutative tensor multiplet with $p_{||} = 2/R_{||}$ is described by quantum mechanics on a blown-up A_1-singularity. In this chapter we will study this quantum mechanics and calculate the low energy scattering, in the quantum mechanics.

The organization is as follows. In section 5.2 we describe the leading order low-energy limit of the noncommutative five-brane theory. That is our motivation for studying the quantum mechanics on a blown-up A_1 singularity. In section 5.3 we describe it in detail. In subsection 5.3.3 we calculate the s-wave scattering amplitude. In section 5.4, we present the calculation for scattering of two scalar particles in field-theory, up to order $O(\Theta)^2$.

5.2 Motivation: noncommutative five-brane

One motivation for studying the QM on the blown-up A_1-singularity is that it is the M(atrix)-model for the noncommutative deformation of a free 5+1D tensor-multiplet. We will now describe this theory at lowest order in the "noncommutativity."

5.2.1 Free five-brane

A single five-brane is described, at low-energies, by a free tensor multiplet of $\mathcal{N} = (2, 0)$ supersymmetry. The bosonic fields are 5 free scalars, ϕ^I $(I = 1 \dots 5)$ and a 3-form tensor field-strength H_{ijk} with equations of motion:

$$H_{ijk} = -\frac{1}{6}\epsilon_{ijklmn}H^{lmn}, \qquad \partial_{[l}H_{ijk]} = 0 \ . \tag{5.5}$$

Here, indices are lowered and raised with the metric:

$$\eta_{mn}dx^m dx^n = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 \ , \tag{5.6}$$

and the anti-symmetric ϵ-symbol is normalized such that: $\epsilon_{012345} = +1$. In particular, we have: $H_{012} = H_{345}$. The notation $[ij \dots k]$ means complete anti-symmetrization. Thus:

$$T_{[i_1 \dots i_n]} \equiv \frac{1}{n!}\sum_{\sigma\in S_n}(-)^\sigma T_{i_{\sigma(1)}\dots i_{\sigma(n)}}.$$

The normalization is such that on 3-cycles:

$$\int H_{ijk}dx_i dx_j dx_k \in 2\pi\mathbb{Z} \ . \tag{5.7}$$

Later on, we will need the propagator:

$$\langle H_{ijk}(x)H_{lmn}(y)\rangle = \int \frac{d^6p}{(2\pi)^6}e^{ip(x-y)}G_{ijk;lmn}(p) \ . \tag{5.8}$$

We calculate it by adding a self-dual part to H_{ijk} and writing it as $H_{ijk} = 3\partial_{[i}B_{jk]}$ with the action (see for instance [52] for details):

$$-\frac{1}{24\pi}\int H_{ijk}H^{ijk}d^6x \ . \tag{5.9}$$

We then keep only the anti-self-dual part of the propagator. The result is:

$$\begin{aligned} G_{ijk;lmn}(p) &= \frac{18\pi}{p^2+i\epsilon}\eta_{rl}\eta_{sm}\eta_{tn}\left(p_{[i}p^{[r}\delta_j^s\delta_{k]}^{t]} - \frac{1}{6}\epsilon_{ijk}{}^{\sigma[st}p^{r]}\right) \\ &\quad +\frac{\pi}{2}\left(\epsilon_{ijklmn} - 6\delta_{[i}^{[r}\delta_j^s\delta_{k]}^{t]}\eta_{lr}\eta_{ms}\eta_{nt}\right) \ . \end{aligned} \tag{5.10}$$

5.2.2 The interacting theory

The interacting theory is described by interactions $L_{int}(\Theta)$ that depend on a constant anti-self-dual 3-form Θ^{ijk}. It satisfies:

$$\Theta_{ijk} = -\frac{1}{6}\epsilon_{ijklmn}\Theta^{lmn} \ . \tag{5.11}$$

L_{int} involves the fields ϕ^I, H_{ijk}, the fermions and their derivatives.

To first order in Θ, the interactions can be described by a self-dual dimension-9 operator, $\mathcal{O}_{ijk}$, in the free theory. The interaction is $\delta L = \Theta^{ijk}\mathcal{O}_{ijk}$.

The bosonic part of the interaction turns out to be:

$$\delta L = \frac{1}{96\pi}\Theta^{ijk}H_{ijl}H^{mnl}H_{mnk} + \frac{1}{4}\Theta^{ikl}H_{jkl}\partial_i\Phi^I\partial^j\Phi^I + O(\Theta)^2 \ . \tag{5.12}$$

Here, H should not be confused with the critical asymptotic value of the tensor field-strength on the five-brane in the construction of the theory from M-theory. The field H is fluctuating and is assumed to go to zero at infinity.

At order $O(\Theta)^2$, the scalar fields have a quartic interaction:

$$-\frac{\pi}{2}\eta_{kn}\Theta^{ijk}\Theta^{lmn}\partial_i\Phi^I\partial_j\Phi^J\partial_l\Phi^I\partial_m\Phi^J \ . \tag{5.13}$$

These terms can be determined by dimensional reduction, as we will now explain.

5.2.3 Dimensional reduction to (4+1)D

If we compactify on $\mathbf{S}^1$ of circumference $2\pi R$, the dimensional reduction to (4+1)D proceeds according to:

$$F_{\mu\nu} = 2\pi R H_{\mu\nu 5}, \qquad \theta^{\mu\nu} = \Theta^{\mu\nu 5}, \qquad g^2 = 4\pi^2 R, \qquad \phi^I = (2\pi)^{3/2} R\Phi^I \ . \tag{5.14}$$

To first order in θ, the 4+1D action is [81, 54]:

$$\begin{aligned} L_{4+1D} &= \frac{1}{2g^2}\partial_\mu\phi^I\partial^\mu\phi^I + \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} \\ &+ \frac{1}{2g^2}\theta^{\mu\nu}F_{\nu\sigma}F^{\sigma\tau}F_{\tau\mu} - \frac{1}{2g^2}\theta^{\mu\nu}F_{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} \\ &+ \frac{1}{g^2}\theta^{\mu\nu}F_{\mu\sigma}\partial_\nu\phi^I\partial^\sigma\phi^I - \frac{1}{4g^2}\theta^{\mu\nu}F_{\mu\nu}\partial_\sigma\phi^I\partial^\sigma\phi^I \ . \end{aligned} \tag{5.15}$$

Assuming that supersymmetry protects the leading interactions from loop-corrections, we can obtain (5.12)-(5.13) by requiring that dimensional reduction should produce the latter corrections in 4+1D SYM.

5.3 QM on a blown-up A_1 singularity

In this section we will describe in detail the quantum mechanics on the target space – the blown up A_1-singularity. The quantum mechanics has $\mathcal{N} = 8$ supersymmetry and this is related to the hyper-Kähler structure on the target space. For a generic description of QM on hyper-Kähler manifolds, see [75].

5.3.1 The geometry

As described in the first section, the M(atrix)-model of k coincident five-branes in the sector with longitudinal momentum $p_{||} = N$ is postulated [4, 89, 20] to be described by quantum mechanics on the manifold $\mathcal{M}_{N,k}$ defined by

$$\left(\begin{array}{ccc} [X, X^\dagger] + [Y, Y^\dagger] + Z^\dagger Z - W^\dagger W & = & \xi^2 Id_{N\times N} \\ [X, Y] + Z^\dagger W & = & 0 \end{array} \right) /U(N) \ , \qquad (5.16)$$

where the group $U(N)$ acts on the $N \times N$ complex matrices X and Y and on the $k \times N$ complex matrices Z and W in the natural way. For any N and k, the trace of matrices X and Y is a flat four-dimensional space $\mathbf{R^4}$ and so we can write $\mathcal{M}_{N,k} = \mathbf{R^4} \times \tilde{\mathcal{M}}_{\mathbf{N,k}}$. This flat four-dimensional part corresponds to the center-of-mass coordinates. We study a single five-brane, $k = 1$, and $N = 2$. The manifold $\tilde{\mathcal{M}}_{N=2,k=1} \equiv \mathcal{M}$ is the blown-up A_1 singularity. It must be the same as $\tilde{\mathcal{M}}_{N=1,k=2}$, as can be confirmed explicitly. The metric is easier to obtain in the second case, for $k = 2, N = 1$, when $\mathcal{M}$ can be embedded in $\mathbb{C}^4$ as

$$\mathrm{Tr}\left(A^\dagger A \sigma_i\right) = \xi^2 \delta_i^3 \qquad\qquad A \sim e^{i\epsilon} A \ , \qquad (5.17)$$

where A is a 2×2 complex matrix and σ_i are the Pauli matrices.

These conditions (ignoring the U(1) quotient for now) can be satisfied by parameterizing A as follows

$$A = \xi e^{i\epsilon} g \begin{pmatrix} \cosh r & 0 \\ 0 & \sinh r \end{pmatrix} , \tag{5.18}$$

where g is an arbitrary element of SU(2), and ξ, r and ϵ are real. The induced metric on this five-(real)dimensional manifold is given by

$$\begin{aligned} \frac{\mathrm{Tr}\left(dA^\dagger dA\right)}{\xi^2} &= \cosh(2r)\left[dr^2 + \frac{1}{2}\mathrm{Tr}(dg^\dagger dg)\right] + \frac{1}{\cosh(2r)}\left[(1\ 0)dg^\dagger g \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]^2 \\ &+ \cosh(2r)\left[d\epsilon + \frac{i}{\cosh(2r)}(1\ 0)dg^\dagger g \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]^2 . \end{aligned} \tag{5.19}$$

To obtain the U(1) quotient, we must choose a function $\epsilon(r,g)$ such that the distance between $(r, g, \epsilon(r,g))$ and $(r+dr, g+dg, \epsilon(r+dr, g+dg))$ is minimized. This corresponds to choosing $\epsilon(r,g)$ in such a way that the last term in the above equation vanishes. Thus, the metric on $\mathcal{M}$ is simply

$$\frac{ds^2|_{\mathcal{M}}}{\xi^2} = \cosh(2r)\left[dr^2 + \frac{1}{2}\mathrm{Tr}(dg^\dagger dg)\right] + \frac{1}{\cosh(2r)}\left[(10)dg^\dagger g \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]^2 . \tag{5.20}$$

It is apparent from equation (5.17) that $\mathcal{M}$ is invariant under SU(2) acting on A on the left and under U(1) acting on the right. g can be parametrized by three angles θ, ϕ and α in such a way as to make these symmetries explicit

$$g = \begin{pmatrix} \cos(\theta/2)e^{i(\alpha-\phi)/2} & -\sin(\theta/2)e^{i(-\alpha-\phi)/2} \\ \sin(\theta/2)e^{i(\alpha+\phi)/2} & \cos(\theta/2)e^{i(-\alpha+\phi)/2} \end{pmatrix} . \tag{5.21}$$

θ runs from 0 to π, and ϕ and α run from 0 to 2π. With this parametrization we find that the metric becomes

$$
\begin{aligned}
ds^2 &= \xi^2 \left\{ \cosh(2r) dr^2 + \frac{\cosh(2r)}{4}(d\theta^2 + \sin^2\theta d\phi^2) \right. \\
&+ \left. \frac{\cosh^2(2r) - 1}{4\cosh(2r)}(d\alpha - \cos\theta d\phi)^2 \right\} .
\end{aligned}
\tag{5.22}
$$

It will be convenient to make the change of variables $\cosh(2r) = R^2$ (R runs from 1 to ∞)

$$
ds^2 = \xi^2 \left\{ \frac{R^4}{R^4 - 1} dR^2 + \frac{R^2}{4}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{R^4 - 1}{4R^2}(d\alpha - \cos\theta d\phi)^2 \right\} . \tag{5.23}
$$

It is apparent that, for large R, $\mathcal{M}$ approaches flat space, namely $\mathbf{R^4}/\mathbb{Z}_\mathbf{2}$. This is the metric for the blown-up A_1 singularity.

For later reference, let us here record that the scalar Laplacian is

$$
\begin{aligned}
\nabla^2 &= \xi^{-2} \left\{ \frac{1}{R^3} \partial_R \left(\frac{R^4 - 1}{R} \partial_R \right) \right. \\
&+ \frac{4}{R^2} \left[\frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta) + \frac{1}{\sin^2\theta} \left(\partial_\phi^2 + \partial_\alpha^2 + 2\cos\theta \partial_\phi \partial_\alpha \right) \right] \\
&+ \left. \frac{4}{R^2(R^4 - 1)} \partial_\alpha^2 \right\} ,
\end{aligned}
\tag{5.24}
$$

and that it is fully separable.

It is known that the manifold $\mathcal{M}$ is a hyper-Kähler manifold, a fact crucial to the existence of $\mathcal{N} = 8$ supersymmetric QM. An explicit hyper-Kähler structure for $\mathcal{M}$ will now be given.

The following

$$
\begin{aligned}
a &\equiv (R^4-1)^{1/4}\sin\frac{\theta}{2}\exp\left(\frac{i}{2}(\alpha+\phi)\right)\\
b &\equiv (R^4-1)^{1/4}\cos\frac{\theta}{2}\exp\left(\frac{i}{2}(\alpha-\phi)\right)
\end{aligned} \tag{5.25}
$$

are good complex coordinates for $\mathcal{M}$. In these coordinates, the metric is

$$
\begin{aligned}
g_{a\bar{a}} &= g_{\bar{a}a} &&= \frac{(a\bar{a}+b\bar{b})^3+b\bar{b}}{(a\bar{a}+b\bar{b})^2\sqrt{(a\bar{a}+b\bar{b})^2+1}}\\
g_{b\bar{b}} &= g_{\bar{b}b} &&= \frac{(a\bar{a}+b\bar{b})^3+a\bar{a}}{(a\bar{a}+b\bar{b})^2\sqrt{(a\bar{a}+b\bar{b})^2+1}}\\
g_{a\bar{b}} &= g_{\bar{b}a} &&= -\frac{\bar{a}b}{(a\bar{a}+b\bar{b})^2\sqrt{(a\bar{a}+b\bar{b})^2+1}}\\
g_{b\bar{a}} &= g_{\bar{a}b} &&= -\frac{a\bar{b}}{(a\bar{a}+b\bar{b})^2\sqrt{(a\bar{a}+b\bar{b})^2+1}}\ ,
\end{aligned} \tag{5.26}
$$

which is clearly Hermitian and easily confirmed to be Kähler, with a Kähler potential

$$
K = X - tan^{-1}X \qquad \text{where} \qquad X = \sqrt{(a\bar{a}+b\bar{b})^2+1}\ . \tag{5.27}
$$

Also, the determinant is $g = 1$, so $\mathcal{M}$ is Ricci-flat, since $R_{i\bar{j}} = -\partial_i\partial_j \ln\det(g)$. The Kähler form, Ω_1, is as usual $(\Omega_1)_{i\bar{j}} = -(\Omega_1)_{\bar{j}i} = ig_{i\bar{j}}$. We have 2 more Kähler forms Ω_2 and Ω_3, satisfying the hyper-Kähler condition $g^{\alpha\gamma}(\Omega_a)_{\alpha\beta}(\Omega_b)_{\gamma\delta} = \epsilon_{abc}(\Omega_c)_{\beta\delta} + \delta_{ab}g_{\beta\delta}$, where $\alpha = i, \bar{j}$. Explicitly, these are (written in the complex coordinates $(a, b, \bar{a}, \bar{b})$)

$$
(\Omega_2)_{\alpha\beta} = \begin{pmatrix} 0 & i & 0 & 0\\ -i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0 \end{pmatrix} \qquad (\Omega_3)_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{pmatrix}\ . \tag{5.28}
$$

Later, we will need a complex structure defined as usual by

$$(\Omega_2)_{\alpha\beta} = -g_{\alpha\gamma} J^{\gamma}_{\beta} \ . \tag{5.29}$$

5.3.2 Quantization

The construction of the $\mathcal{N} = 8$ supersymmetric algebra and its representation in a coordinate basis presented in this section is general to any 2-(complex)dimensional hyper-Kähler manifold. We will first construct the $\mathcal{N} = 4$ supersymmetric quantum algebra, following a procedure similar to that in [42], and then extent the supersymmetry to $\mathcal{N} = 8$.

The Lagrangian is obtained by dimensionally reducing the d=4, $\mathcal{N} = 1$ chiral supergravity Lagrangian in flat space [117]. Since after dimensional reduction the original SO(3,1) spinor indices lose their meaning, we will use SU(2) spinors, making no distinction between dotted and undotted spinor indices. Writing SU(2) spinors on the right and SO(3,1) spinors on the left (using the spinor conventions in [117]) we define

$$\chi^\alpha \equiv \chi^\alpha \ , \qquad \bar{\chi}^\alpha \equiv -\bar{\chi}_{\dot\alpha} = \bar{\sigma}^{0\dot\alpha\alpha}\bar{\chi}_{\dot\alpha} \ . \tag{5.30}$$

The spinor products can be defined in our conventions as

$$\begin{aligned}
\psi\chi &\equiv \epsilon_{\alpha\beta}\psi^\alpha\chi^\beta = \psi^\alpha\chi_\alpha \\
\bar{\psi}\chi &\equiv \epsilon_{\alpha\beta}\bar{\psi}^\alpha\chi^\beta = \bar{\psi}^\alpha\chi_\alpha \\
\bar{\psi}\bar{\chi} &\equiv \epsilon_{\alpha\beta}\bar{\psi}^\alpha\bar{\chi}^\beta = \bar{\psi}^\alpha\bar{\chi}_\alpha \ .
\end{aligned} \tag{5.31}$$

Notice that this implies that $(\chi^\alpha)^\dagger = \bar{\chi}_\alpha$ and $(\chi_\alpha)^\dagger = -\bar{\chi}^\alpha$ which in turn gives

$$(\chi\phi)^\dagger = -\bar{\chi}\bar{\phi} \qquad (\bar{\chi}\bar{\phi})^\dagger = -\chi\phi \qquad (\bar{\chi}\phi)^\dagger = \bar{\chi}\phi \ . \tag{5.32}$$

Finally, following [117], $\epsilon^{12} = -\epsilon_{12} = 1$.

In this notation, the dimensionally reduced Lagrangian is

$$\begin{aligned} \mathcal{L} &= g_{i\bar{j}}\dot{z}^i\dot{\bar{z}}^{\bar{j}} - ig_{i\bar{j}}\bar{\chi}^{\bar{j}}(D_t\chi^i) - \frac{1}{4}R_{i\bar{j}k\bar{l}}(\chi^i\chi^k)(\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}}) \\ &\sim g_{i\bar{j}}\dot{z}^i\dot{\bar{z}}^{\bar{j}} - \frac{i}{2}g_{i\bar{j}}\bar{\chi}^{\bar{j}}(D_t\chi^i) + \frac{i}{2}g_{i\bar{j}}(D_t\bar{\chi}^{\bar{j}})\chi^i - \frac{1}{4}R_{i\bar{j}k\bar{l}}(\chi^i\chi^k)(\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}}) \ . \end{aligned} \tag{5.33}$$

where $D_t\chi^i = \dot{\chi} + \Gamma^i_{ab}\chi^a\dot{z}^b$ and the two forms of the Lagrangian differ only by a total time derivative. We will use the second form, which is real.

The conjugate momenta are

$$\begin{aligned} P_i &\equiv \frac{\partial\mathcal{L}}{\partial z^i} = g_{i\bar{j}}\dot{\bar{z}}^{\bar{j}} - \frac{i}{2}\partial_k g_{i\bar{j}}\bar{\chi}^{\bar{j}}\chi^k \\ \bar{P}_{\bar{j}} &\equiv \frac{\partial\mathcal{L}}{\partial\bar{z}^{\bar{j}}} = g_{i\bar{j}}\dot{z}^i + \frac{i}{2}\partial_{\bar{l}}g_{i\bar{j}}\bar{\chi}^{\bar{l}}\chi^i \\ \pi_{i\alpha} &\equiv \frac{\partial\mathcal{L}}{\partial\chi^{i\alpha}} = -\frac{i}{2}g_{i\bar{j}}\bar{\chi}^{\bar{j}}_\alpha \\ \bar{\pi}_{\bar{j}\alpha} &\equiv \frac{\partial\mathcal{L}}{\partial\bar{\chi}^{\bar{j}\alpha}} = +\frac{i}{2}g_{i\bar{j}}\chi^i_\alpha \ , \end{aligned} \tag{5.34}$$

and their canonical Poisson brackets are

$$\begin{aligned} \{z^k, P_i\} &= \delta^i_k & \{\bar{z}^{\bar{j}}, \bar{P}_{\bar{l}}\} &= \delta^{\bar{j}}_{\bar{l}} \\ \{\chi^{i\alpha}, \pi_{k\beta}\} &= -\delta^i_k\delta^\alpha_\beta & \{\bar{\chi}^{\bar{j}\alpha}, \bar{\pi}_{\bar{l}\beta}\} &= -\delta^{\bar{j}}_{\bar{l}}\delta^\alpha_\beta \ . \end{aligned} \tag{5.35}$$

This system possesses fermionic constraints of the second kind

$$\phi_i^\alpha \equiv \pi_i^\alpha + \frac{i}{2} g_{i\bar{j}} \bar{\chi}^{\bar{j}\alpha} = 0 \qquad \bar{\phi}_{\bar{j}\alpha} \equiv \bar{\pi}_{\bar{j}\alpha} - \frac{i}{2} g_{i\bar{j}} \chi_\alpha^i = 0 \;, \tag{5.36}$$

whose Poisson bracket is

$$\{\phi_i^\alpha, \bar{\phi}_{\bar{j}\beta}\} = -i g_{i\bar{j}} \delta_\beta^\alpha \;. \tag{5.37}$$

Following standard procedure [28], we define the Dirac brackets

$$\{\circ,\circ\}_D \equiv \{\circ,\circ\} - \{\circ,\phi_i^\alpha\} \frac{1}{\{\phi_i^\alpha, \bar{\phi}_{\bar{j}\beta}\}} \{\bar{\phi}_{\bar{j}\beta},\circ\} - \{\circ,\bar{\phi}_{\bar{j}\beta}\} \frac{1}{\{\bar{\phi}_{\bar{j}\beta}, \phi_i^\alpha\}} \{\phi_i^\alpha,\circ\} \;. \tag{5.38}$$

Canonical quantization proceeds by substituting $\{\circ,\circ\}_D \to -i[\circ,\circ]$ or $-i\{\circ,\circ\}$, as appropriate. Evaluating the Dirac brackets for all quantities, we obtain the following algebra

$$[z^k, P_i] = i\delta_i^k \qquad [\bar{z}^{\bar{l}}, \bar{P}_{\bar{j}}] = i\delta_{\bar{j}}^{\bar{l}} \qquad \{\chi^{i\alpha}, \bar{\chi}_\beta^{\bar{j}}\} = g^{i\bar{j}} \delta_\beta^\alpha \;. \tag{5.39}$$

The commutators of z_i and $\bar{z}_{\bar{j}}$ with the fermions $\chi^{i\alpha}$ and $\bar{\chi}^{\bar{j}\beta}$ vanish. Define

$$K_i \equiv \frac{i}{2} \partial_k g_{i\bar{j}} \bar{\chi}^{\bar{j}\beta} \chi_\beta^k \qquad \bar{K}_{\bar{j}} \equiv -\frac{i}{2} \partial_{\bar{l}} g_{i\bar{j}} \bar{\chi}^{\bar{l}\beta} \chi_\beta^i \;. \tag{5.40}$$

The commutation relationships of P_k and $\bar{P}_{\bar{j}}$ alone are not as relevant to what will follow as the commutation relationships involving $P + K$ and $\bar{P} + \bar{K}$. With some work, it can be shown that

$$\begin{aligned} [(P+K)_k, \chi^{i\alpha}] &= i\Gamma_{ka}^i \chi^{a\alpha} & [(P+K)_k, \bar{\chi}^{\bar{j}\alpha}] &= 0 \\ [(\bar{P}+\bar{K})_{\bar{l}}, \chi^{i\alpha}] &= 0 & [(\bar{P}+\bar{K})_{\bar{l}}, \bar{\chi}^{\bar{j}\alpha}] &= i\bar{\Gamma}_{\bar{l}\bar{b}}^{\bar{j}} \bar{\chi}^{\bar{b}\alpha} \end{aligned} \tag{5.41}$$

and that

$$[(P+K)_i,(P+K)_k]=[(\bar{P}+\bar{K})_{\bar{j}},(\bar{P}+\bar{K})_{\bar{l}}] \;=\; 0 \qquad (5.42)$$
$$[(P+K)_i,(\bar{P}+\bar{K})_{\bar{j}}] \;=\; -R_{i\bar{j}a\bar{b}}\chi^a\bar{\chi}^{\bar{b}}\ .$$

The (classical) Hamiltonian obtained by the standard procedure is

$$H=g^{i\bar{j}}(P+K)_i(\bar{P}+\bar{K})_{\bar{j}}+\frac{1}{4}R_{i\bar{j}a\bar{b}}(\chi^i\chi^a)(\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{b}}) \qquad (5.43)$$

Since $R_{a\bar{b}}=g^{i\bar{j}}R_{i\bar{j}a\bar{b}}=0$, there are no ordering ambiguities and thus (5.43) can be simply taken to be the quantum Hamiltonian. A very important cyclic identity for the fermionic coordinates is that

$$\chi^\alpha(\psi\phi)+\psi^\alpha(\phi\chi)+\phi^\alpha(\chi\psi)=0\ . \qquad (5.44)$$

This identity must be used with care for the quantum fermionic coordinates, since they have non-trivial anti-commutation relationships. Nevertheless, the Hamiltonian can be rewritten as

$$H=g^{i\bar{j}}(p+K)_i(\bar{p}+\bar{K})_{\bar{j}}-\frac{1}{2}R_{i\bar{j}a\bar{b}}(\chi^i\bar{\chi}^{\bar{j}})(\chi^a\bar{\chi}^{\bar{b}})\ . \qquad (5.45)$$

The (classical) supersymmetry transformations are given by [117]

$$\delta_\xi z^i=\xi\chi^i \qquad \delta_\xi\chi^{i\alpha}=-i\bar{\xi}^\alpha\dot{z}^i-\Gamma^i_{jk}(\xi\chi^j)\chi^{k\alpha}\ . \qquad (5.46)$$

With some work, the Nöether current can be calculated for the real form of the

Lagrangian (5.33) to be

$$J_{SUSY} = g_{i\bar{j}}\dot{\bar{z}}^{\bar{j}}\xi\chi^i - g_{i\bar{j}}\dot{z}^i\bar{\xi}\bar{\chi}^{\bar{j}} \ . \tag{5.47}$$

The minus sign is related to the minus sign in equation (5.32). Defining the SUSY generators through $J_{SUSY} = Q^\alpha\xi_\alpha + \bar{Q}_\alpha\bar{\xi}^\alpha$ we obtain that

$$Q^\alpha = g_{i\bar{j}}\dot{\bar{z}}^{\bar{j}}\chi^{i\alpha} = (P+K)_i\chi^{i\alpha} \qquad \bar{Q}_\alpha = g_{i\bar{j}}\dot{z}^i\bar{\xi}\bar{\chi}^{\bar{j}} = (\bar{P}+\bar{K})_{\bar{j}}\bar{\chi}^{\bar{j}}_\alpha \ . \tag{5.48}$$

There are no ordering ambiguities when interpreting these as quantum operators.

It is now a matter of another careful computation to confirm that the Q's satisfy the SUSY algebra

$$\{Q^\alpha, Q^\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \qquad \{Q^\alpha, \bar{Q}_\beta\} = \delta^\alpha_\beta H \ . \tag{5.49}$$

We must represent this algebra in a coordinate basis. There are 4 fermionic coordinates and thus the Hilbert space can be written as a $2^4 = 16$-dimensional vector of complex functions of z and $\bar{z}$. Equation (5.39) suggests that we associate P_i with $-i\partial_i$ and $\bar{P}_{\bar{j}}$ with $-i\partial_{\bar{j}}$. Naively, this would disagree with (5.41) and (5.43). We need to pay attention to how these partial derivatives act on the fermionic basis.

Let us begin by noticing that

$$[(P+K)_k, \chi^\alpha_{\bar{j}}] = [(P+K)_k, \bar{\chi}^{\bar{j}\alpha}] = 0 \qquad \{\chi^\alpha_{\bar{l}}, \bar{\chi}^{\bar{j}}_\beta\} = \delta^{\bar{j}}_{\bar{l}}\delta^\alpha_\beta \ . \tag{5.50}$$

and

$$[(\bar{P}+\bar{K})_{\bar{l}}, \chi^{i\alpha}] = [(\bar{P}+\bar{K})_{\bar{l}}, \bar{\chi}^\alpha_i] = 0 \qquad \{\chi^{i\alpha}, \bar{\chi}_{k\beta}\} = \delta^i_k\delta^\alpha_\beta \ . \tag{5.51}$$

Define $|\downarrow\rangle$ by $\bar{\chi}^{\bar{j}}_{\beta}|\downarrow\rangle = \bar{\chi}_{i\beta}|\downarrow\rangle = 0$. It is consistent with all commutation relationships to write that $(P+K)_i[f(z,\bar{z})|\downarrow\rangle] = (-i\partial_i f(z,\bar{z}))|\downarrow\rangle$ and $(\bar{P}+\bar{K})_{\bar{j}}[f(z,\bar{z})|\downarrow\rangle] = -i\partial_{\bar{j}} f(z,\bar{z})|\downarrow\rangle$. From the no-fermions state $|\downarrow\rangle$ we can construct the one-fermion state in two ways: using $\chi^{i\alpha}$ or $\chi^{\alpha}_{\bar{j}}$. Let $|\chi^{i\alpha}\rangle \equiv \chi^{i\alpha}|\downarrow\rangle$ and $|\chi^{\alpha}_{\bar{j}}\rangle \equiv \chi^{\alpha}_{\bar{j}}|\downarrow\rangle = g_{i\bar{j}}\chi^{i\alpha}|\downarrow\rangle$. It can be then checked by an explicit calculation that the following is consistent with all commutation relationships

$$\begin{aligned}(P+K)_i[f^{\bar{j}}_{\alpha}(z,\bar{z})|\chi^{\alpha}_{\bar{j}}\rangle] &= (-i\partial_i f^{\bar{j}}_{\alpha}(z,\bar{z}))|\chi^{\alpha}_{\bar{j}}\rangle \\ (\bar{P}+\bar{K})_{\bar{j}}[f_{i\alpha}(z,\bar{z})|\chi^{i\alpha}\rangle] &= (-i\partial_{\bar{j}} f_{i\alpha}(z,\bar{z}))|\chi^{i\alpha}\rangle \ .\end{aligned} \tag{5.52}$$

Thus, one should think of $|\downarrow\rangle$ as being independent of both z and $\bar{z}$, of $\chi^{i\alpha}$ and $\bar{\chi}^{\alpha}_{i}$ as being holomorphic functions and $\chi^{\alpha}_{\bar{j}}$ and $\bar{\chi}^{\bar{j}\alpha}$ as being antiholomorphic functions.

We can continue this procedure by defining two, three and four-fermion states $|\chi\chi\rangle$, $|\chi\chi\chi\rangle$ and $|\chi\chi\chi\chi\rangle$, and checking each time that the commutation relationships work. The computation is made easier if one notices that the four-fermion state $|\chi\chi\chi\chi\rangle$ is the same as $|\uparrow\rangle$ defined by $\chi^{\beta}_{\bar{j}}|\uparrow\rangle = \chi^{i\beta}|\uparrow\rangle = 0$ and that $|\chi\chi\chi\rangle$ can be written as $|\bar{\chi}\rangle \equiv \bar{\chi}|\uparrow\rangle$. With this explicit construction, we can write the energy eigenvalue equation $H|\rangle = E|\rangle$ as a differential equation. Notice that H commutes with the fermion number operator $\chi^i\bar{\chi}_i$ and so we can write different differential equations for each fermion number. For the no-fermions state, $f|\downarrow\rangle$, $H|\rangle = E|\rangle$ can be written as

$$-g^{i\bar{j}}\partial_i\partial_{\bar{j}} f(z,\bar{z}) = Ef(z,\bar{z}) \ . \tag{5.53}$$

For the one-fermion states, $f_{i\alpha}(z,\bar{z})|\chi^{i\alpha}\rangle$, the equation is

$$-g^{k\bar{j}}\partial_k\partial_{\bar{j}} f_{i\alpha} + g^{k\bar{j}}\Gamma^{a}_{ik}\partial_{\bar{j}} f_{a\alpha} = Ef_{i\alpha} \ . \tag{5.54}$$

And, finally, for the two-fermion states, $f_{nm\alpha\beta}(z,\bar{z})|\chi^{n\alpha}\chi^{m\beta}\rangle$, we obtain

$$-g^{k\bar{j}}\partial_k\partial_{\bar{j}}f_{nm\alpha\beta} + g^{k\bar{j}}(\Gamma^a_{nk}\partial_{\bar{j}}f_{am\alpha\beta} + \Gamma^a_{mk}\partial_{\bar{j}}f_{na\alpha\beta}) - R_{n\bar{b}m\bar{d}}g^{a\bar{b}}g^{c\bar{d}}f_{ac\alpha\beta} = Ef_{nm\alpha\beta}\ . \qquad (5.55)$$

The equations for three- and four-fermion states can be obtained by analogy to (5.54) and (5.53).

Given the tensor J defined in equation (5.29), we obtain four more supersymmetry generators

$$S^\alpha \equiv (\bar{P}+\bar{K})_{\bar{j}}J^{\bar{j}}_i\chi^{i\alpha} \qquad \bar{S}_\alpha \equiv (P+K)_iJ^i_{\bar{j}}\bar{\chi}^{\bar{j}}_\alpha\ . \qquad (5.56)$$

These can be confirmed to satisfy $\mathcal{N}=8$ SUSY algebra, namely

$$\{Q^\alpha,\bar{Q}_\beta\} = \{S^\alpha,S_\beta\} = \delta^\alpha_\beta H \qquad (5.57)$$

and

$$\begin{aligned}\{Q^\alpha,\bar{S}_\beta\} = \{S^\alpha,\bar{Q}_\beta\} = \{Q^\alpha,Q^\beta\} = \{S^\alpha,Q^\beta\} &= \\ \{S^\alpha,S^\beta\} = \{\bar{Q}_\alpha,\bar{Q}_\beta\} = \{\bar{S}_\alpha,\bar{Q}_\beta\} = \{\bar{S}_\alpha,\bar{S}_\beta\} &= 0\ .\end{aligned} \qquad (5.58)$$

In proving the above relationships, we need to use the properties of J: the metric is hermitian, J is covariantly constant and the Nijenhuis tensor, $\mathcal{N}(J)$, vanishes.

5.3.3 Scattering

Consider the simplest scattering question - s-wave scattering of one of the scalar particles in the low energy regime. This is described by equation (5.53), $-\nabla^2 f = Ef$. The Laplacian was given in equation (5.24). For s-wave scattering, $f = f(R)$ and the differential equation to be solved is

$$-\frac{1}{R^3}\partial_R\left(\frac{R^4-1}{R}\partial_R f(R)\right) = \xi^2 E f(R)\ . \tag{5.59}$$

Let $y^4 = \xi^2 E \ll 1$ and define $x = yR$. We can then rewrite (5.59) as

$$-\frac{1}{x^3}\partial_x\left(\frac{x^4-y^4}{x}\partial_x f(x)\right) = y^2 f(x)\ . \tag{5.60}$$

For $x \ll 1/y$ we can treat the RHS of equation (5.60) as a small perturbation. Keeping the solution finite at x=y, we obtain a solution as an expansion in y

$$1-\frac{1}{8}x^2y^2+\frac{1}{192}x^4y^4-\frac{1}{9216}x^6y^6+o(y^8)\ . \tag{5.61}$$

For $x \gg y$, rewrite (5.60) as

$$-\frac{1}{x^3}\partial_x\left(x^3\partial_x f(x)\right) - y^2 f(x) = -y^4\frac{1}{x^3}\partial_x\left(\frac{1}{x}\partial_x f(x)\right)\ . \tag{5.62}$$

The RHS is again a small perturbation. The solution with the RHS set to zero is the asymptotic solution at infinity

$$\frac{2}{y}\frac{J_1(yx)+aY_1(yx)}{x} \tag{5.63}$$

(the factor $(2/y)$ is for convenience only). We need to obtain the lowest order correction due to the RHS, which is $-y^6/24x^2$. The expansion in y is thus

$$1-\frac{1}{8}x^2y^2+\frac{1}{192}x^4y^4-\left(\frac{1}{9216}x^6+\frac{1}{24x^2}\right)y^6+o(y^8)-\frac{4a}{\pi y^2x^2}+... \tag{5.64}$$

Since the region of validity of the two expansions (5.61) and (5.64) in y overlaps for

$x \sim 1$, they should match. We need to cancel the x^{-2} terms in expansion (5.64), and thus

$$a = -\frac{\pi y^8}{96} = -\frac{\pi \xi^4 E^2}{96} \ . \tag{5.65}$$

5.3.4 The $SO(5)$ symmetry and two-particle states

The tensor multiplet on the five-brane contains 5 real scalars with an $SO(5)$ symmetry. This symmetry should somehow be visible in the M(atrix)-model we have just constructed. In this section we will find the $SO(5)$ symmetry which commutes with the Hamiltonian and construct a real five-dimensional representation of it.

The $SO(5)$ symmetry we are looking for contains the $SO(3)_\Omega$ symmetry acting naturally on the three Kähler forms Ω_i's, as well as the $SU(2) = SO(3)_f$ symmetry acting on the fermionic indices. Since the $SO(5)$ symmetry is connected to the scalars, it should act non-trivially on the states $f|\downarrow\rangle$, $f|\uparrow\rangle$ and $|\chi^{n\alpha}\chi^{m\beta}\rangle$. Let us see what is the action of the $SO(3)_\Omega$ symmetry on the fermion vacuum states. For example, consider using $\Omega_1 + \epsilon\Omega_2$ instead of Ω_1 to define the complex coordinates $(z^i, \bar{z}^{\bar{j}})$. The required change of coordinates is

$$dz^i \to dz^i + \epsilon g^{i\bar{j}}(\Omega_2)_{\bar{j}\bar{k}} d\bar{z}^{\bar{k}} \qquad\qquad d\bar{z}^{\bar{i}} \to d\bar{z}^{\bar{i}} + \epsilon g^{\bar{i}j}(\Omega_2)_{jk} dz^k \ . \tag{5.66}$$

This being just a change of variables, we know how it will act on the fermionic variables $\chi^{i\alpha}$

$$\chi^{i\alpha} \to \chi^{i\alpha} + \epsilon g^{i\bar{j}}(\Omega_2)_{\bar{j}\bar{k}} \bar{\chi}^{\bar{k}}_\alpha \qquad\qquad \bar{\chi}^{\bar{i}}_\alpha \to \bar{\chi}^{\bar{i}}_\alpha + \epsilon g^{\bar{i}j}(\Omega_2)_{jk} \chi^{k\alpha} \ . \tag{5.67}$$

The new $\bar{\chi}$'s must annihilate the new vacuum, and so we infer that

$$|\downarrow\rangle \rightarrow |\downarrow\rangle - \frac{1}{2}\epsilon(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} + \chi^{i2}\chi^{j2}\right)|\downarrow\rangle \ . \tag{5.68}$$

where $1, 2$ are concrete fermionic indices. Similarly, we obtain the action on $|\uparrow\rangle$

$$|\uparrow\rangle \rightarrow |\uparrow\rangle - \frac{1}{2}\epsilon(\Omega_2)_{\bar{i}\bar{j}}\left(\bar{\chi}^{\bar{i}1}\bar{\chi}^{\bar{j}1} + \bar{\chi}^{\bar{i}2}\bar{\chi}^{\bar{j}2}\right)|\uparrow\rangle \ , \tag{5.69}$$

which can be rewritten as

$$|\uparrow\rangle \rightarrow |\uparrow\rangle - \frac{1}{2}\epsilon(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} + \chi^{i2}\chi^{j2}\right)|\downarrow\rangle \ , \tag{5.70}$$

if we define that

$$|\uparrow\rangle \equiv \chi^{11}\chi^{21}\chi^{12}\chi^{22}|\downarrow\rangle \ . \tag{5.71}$$

We also need

$$(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} + \chi^{i2}\chi^{j2}\right)|\downarrow\rangle \rightarrow (\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} + \chi^{i2}\chi^{j2}\right)|\downarrow\rangle + 4\epsilon\left(|\uparrow\rangle + |\downarrow\rangle\right) \ . \tag{5.72}$$

Repeating this calculation for the change of coordinates generated by using $\Omega_1 + \epsilon\Omega_3$ instead of Ω_1,we obtain that the following three states form a basis for a real three dimensional representation of $SO(3)_\Omega$

$$\begin{gathered}\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)\\ \frac{1}{2\sqrt{2}}(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} + \chi^{i2}\chi^{j2}\right)|\downarrow\rangle\\ \frac{i}{\sqrt{2}}\left(|\downarrow\rangle - |\uparrow\rangle\right) \ .\end{gathered} \tag{5.73}$$

The first and third states are singlets under $SO(3)_f$, but the second one is not - it is part of a triplet. Filling in the triplet gives us the complete basis for the $SO(5)$ representation we were after

$$
\begin{aligned}
|\phi^0\rangle &\equiv \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \\
|\phi^1\rangle &\equiv \frac{i}{2\sqrt{2}}(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} - \chi^{i2}\chi^{j2}\right)|\downarrow\rangle \\
|\phi^2\rangle &\equiv \frac{1}{2\sqrt{2}}(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j1} + \chi^{i2}\chi^{j2}\right)|\downarrow\rangle \\
|\phi^3\rangle &\equiv \frac{-i}{2\sqrt{2}}(\Omega_2)_{ij}\left(\chi^{i1}\chi^{j2} + \chi^{i2}\chi^{j1}\right)|\downarrow\rangle \\
|\phi^4\rangle &\equiv \frac{i}{\sqrt{2}}\left(|\downarrow\rangle - |\uparrow\rangle\right) \ .
\end{aligned} \tag{5.74}
$$

By construction the $SO(5)$ transformations must commute with the Hamiltonian. This manifests itself in the fact that

$$
H\left(f_i(z,\bar{z})|\phi^i\rangle\right) = \left(-\nabla^2 f_i(z,\bar{z})\right)|\phi^i\rangle \ , \tag{5.75}
$$

as can be checked explicitly. All five of the scalars have the same equation of motion. We now know which of the 6 two-fermion states $|\chi\chi\rangle$ correspond to scalars and which to the 3-form field. The later is represented by the singlet states under $SO(3)_f$, namely by $f_{ij}\epsilon_{\alpha\beta}|\chi^{i\alpha}\chi^{j\beta}\rangle$ where $f_{ij} = f_{ji}$ has three independent components as needed.

5.3.5 Relation to the M(atrix)-model

The calculation that we performed should be compared with the $O(\Theta)^2$ scattering in the M(atrix)-model of the noncommutative tensor multiplet. We will not perform the comparison in detail here, but only sketch the general picture. For the scattering of two massless particles, there are 4 Feynman diagrams that contribute at $O(\Theta)^2$. One is from the quartic vertex (5.13) and the other contributions are from the tree diagrams (the s, t, u channels) with two cubic $\Phi\Phi H$ vertices as in (5.12) and an $[HH]$ propagator.

Let us see what does this all mean for scattering of two scalar particles on the five-brane. Let the fermionic coordinates corresponding to the center-of-mass coordinates be denoted by $\psi^{i\alpha}$, and those corresponding to the individual particles be $A^{i\alpha} = (\chi^{i\alpha} + \psi^{i\alpha})/2$ and $B^{i\alpha} = (\chi^{i\alpha} - \psi^{i\alpha})/2$. Since for scattering we are only interested in asymptotic states, we can assume that everything happens on a flat manifold. Now, a state of two identical scalars A and B is just $|\phi^0\rangle_A |\phi^0\rangle_B$ where we chose a specific direction under the SO(5). This can be rewritten in terms of the separated coordinates χ and ϕ

$$\begin{aligned} |\phi^0\rangle_A |\phi^0\rangle_B &= \frac{1}{2}\left[|\downarrow\rangle_A|\downarrow\rangle_B + |\uparrow\rangle_A|\uparrow\rangle_B + (|\downarrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B)\right] \\ &= \frac{1}{2}\left(|\downarrow\rangle_\phi|\downarrow\rangle_\chi + |\uparrow\rangle_\phi|\uparrow\rangle_\chi\right) + \sum |\rangle_\phi |\text{scalar}\rangle_\chi \\ &+ \sum |\rangle_\phi |\text{3-form}\rangle_\chi \ . \end{aligned} \tag{5.76}$$

Choosing another-two particle state, we obtain

$$\begin{aligned} |\phi^4\rangle_A |\phi^4\rangle_B &= -\frac{1}{2}\left[|\downarrow\rangle_A|\downarrow\rangle_B + |\uparrow\rangle_A|\uparrow\rangle_B - (|\downarrow\rangle_A|\uparrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B)\right] \\ &= -\frac{1}{2}\left(|\downarrow\rangle_\phi|\downarrow\rangle_\chi + |\uparrow\rangle_\phi|\uparrow\rangle_\chi\right) + \sum |\rangle_\phi |\text{scalar}\rangle_\chi \\ &+ \sum |\rangle_\phi |\text{3-form}\rangle_\chi \ . \end{aligned} \tag{5.77}$$

Thus, the scattering matrix for $\psi^I\psi^I \to \psi^J\psi^J$ will have a form

$${}_A\langle\phi^4| \ {}_B\langle\phi^4| \ S \ |\phi^0\rangle_A \ |\phi^0\rangle_B = -\frac{1}{2}\sigma_1 + \sigma_2 + \sigma_3 \tag{5.78}$$

and that for $\psi^I\psi^I \to \psi^I\psi^I$ will be

$$ {}_A\langle\phi^4|\ {}_B\langle\phi^4|\ S\ |\phi^0\rangle_A\ |\phi^0\rangle_B = \frac{1}{2}\sigma_1 + \sigma_2 + \sigma_3\ , \tag{5.79} $$

where

$$ \begin{aligned} \sigma_1 &= {}_\chi\langle\downarrow|S|\downarrow\rangle_\chi = {}_\chi\langle\uparrow|S|\uparrow\rangle_\chi \\ \sigma_2 &= \left(\sum{}_\chi\langle\text{scalar}|\right) S \left(\sum|\text{scalar}\rangle_\chi\right) \\ \sigma_3 &= \left(\sum{}_\chi\langle\text{3-form}|\right) S \left(\sum|\text{3-form}\rangle_\chi\right)\ , \end{aligned} \tag{5.80} $$

and where we have ignored the (trivial) evolution of the center-of-mass coordinates.

The answers are as we would expect: the difference (equal to σ_1) between the matrix element for $\psi^I\psi^I \to \psi^I\psi^I$ and for $\psi^I\psi^I \to \psi^J\psi^J$ arises to the lowest order in the effective theory from the extra four-scalar vertex $\psi^I\psi^I\psi^J\psi^J$ which is zero for $I = J$. The other two pieces, σ_2 and σ_3 will have different momentum behavior, as can be seen in an explicit Feynman diagram computation.

It is interesting to see what happens in the approximation of a large impact parameter. From the M(atrix)-model we expect a force that behaves as v^2/r^4 where r is the distance between the particles and v is the relative transverse velocity. In the large impact parameter approximation, the t-channel dominates. From (5.10) we see that the H-propagator behaves as $\frac{1}{r^4}$ when there is no longitudinal momentum transfer. The two $\Phi\Phi H$ vertices should contribute a v^2.

5.4 Scattering in the field theory

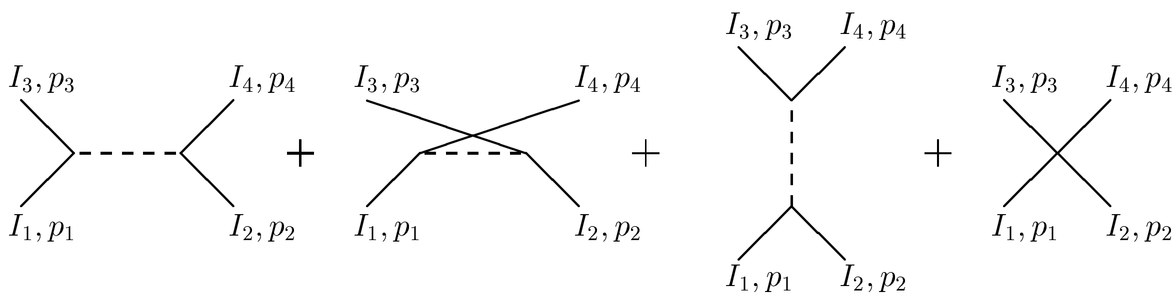
In this section we will describe how to calculate the scattering in field theory. Let us consider the scattering of two scalars to lowest nontrivial order. There are two contributions. One from the $O(\Theta)^2$ quartic vertex and one from a tree diagram with an H-field exchanged. Let us consider the amplitude

$$A^{I_1 I_2 I_3 I_4}(p_1, p_2, p_3, p_4) \ , \tag{5.81}$$

with $p_1 + p_2 = p_3 + p_4$.

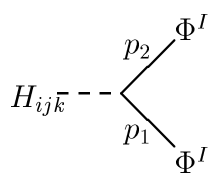
The amplitude is a sum of s, t, u channels and a quartic vertex:

$$A = A_t + A_u + A_s + A_q \ . \tag{5.82}$$

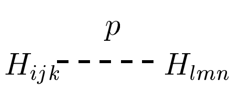


5.4.1 The Feynman rules

The Feynman rules are as follows.



$$= \quad \Theta^{l[ij}(p_2)^{k]}(p_1)_l + \Theta^{l[ij}(p_1)^{k]}(p_2)_l$$



$$= \quad \frac{36\pi}{p^2+i\epsilon}\eta_{rl}\eta_{sm}\eta_{tn}\left(p_{[i}p^{[r}\delta^s_j\delta^{t]}_{k]} - \frac{1}{6}p_u\epsilon_{ijk}{}^{u[st}p^{r]}\right) + \frac{\pi}{2}\left(\epsilon_{ijklmn} - 6\delta^{[r}_{[i}\delta^s_j\delta^{t]}_{k]}\eta_{lr}\eta_{ms}\eta_{nt}\right)$$

The propagator is:

$$\langle H_{ijk}(p)H_{lmn}(-p')\rangle = \frac{36\pi}{p^2+i\epsilon}\eta_{rl}\eta_{sm}\eta_{tn}\left(p_{[i}p^{[r}\delta_j^s\delta_{k]}^{t]} - \frac{1}{6}p_u\epsilon_{ijk}{}^{u[st}p^{r]}\right)\delta^{(6)}(p-p')$$
$$+\frac{\pi}{2}\left(\epsilon_{ijklmn} - 6\delta_{[i}^{[r}\delta_j^s\delta_{k]}^{t]}\eta_{lr}\eta_{ms}\eta_{nt}\right)\delta^{(6)}(p-p') \ . \qquad (5.83)$$

The cubic vertex is:

$$\Lambda_{I_1I_2}^{ijk}(p_1,p_2) = \frac{1}{96\pi}\delta_{I_1I_2}\Theta^{l[ij}(p_2)^{k]}(p_1)_l + \frac{1}{96\pi}\delta_{I_1I_2}\Theta^{l[ij}(p_1)^{k]}(p_2)_l \ . \qquad (5.84)$$

In terms of $p \equiv p_1 + p_2$ and $q \equiv p_1 - p_2$, we can write this as:

$$\Lambda_{I_1I_2}^{ijk}(p,q) = \frac{1}{192\pi}\delta_{I_1I_2}\Theta^{l[ij}p^{k]}p_l + \frac{1}{192\pi}\delta_{I_1I_2}\Theta^{l[ij}q^{k]}q_l \ . \qquad (5.85)$$

The quartic interaction is:

$$A_q = \eta_{pq}\Theta^{klp}\Theta^{ijq}p_{1k}p_{2i}p_{3l}p_{4j}(\delta_{I_1I_4}\delta_{I_2I_3} - \delta_{I_1I_2}\delta_{I_3I_4})$$
$$+\eta_{pq}\Theta^{klp}\Theta^{ijq}p_{1k}p_{2i}p_{3j}p_{4l}(\delta_{I_1I_3}\delta_{I_2I_4} - \delta_{I_1I_2}\delta_{I_3I_4})$$
$$+\eta_{pq}\Theta^{klp}\Theta^{ijq}p_{1k}p_{2l}p_{3i}p_{4j}(\delta_{I_1I_4}\delta_{I_2I_3} - \delta_{I_1I_3}\delta_{I_2I_4}) \ . \qquad (5.86)$$

Let us define:

$$p \equiv p_1 + p_2, \qquad q \equiv p_1 - p_2, \qquad r \equiv p_3 - p_4 \ . \qquad (5.87)$$

We have:

$$p^2 = -q^2 = -r^2, \qquad p\cdot q = p\cdot r = 0 \ . \qquad (5.88)$$

We get $A_s = A'_s \delta_{I_1 I_2} \delta_{I_3 I_4}$ with

$$\begin{aligned} A'_s \;=\; & +\frac{\pi}{2}\eta_{cf}\Theta^{abc}\Theta^{def}p_a r_b p_d r_e + \frac{\pi}{4}\eta_{cf}\Theta^{abc}\Theta^{def}p_a q_b p_d q_e \\ & +\frac{\pi}{4}p^2\eta_{be}\eta_{cf}\Theta^{abc}\Theta^{def}p_a p_d - \frac{\pi}{16}r_l q^l \eta_{be}\eta_{cf}\Theta^{abc}\Theta^{def}q_a r_d \\ & +\frac{\pi}{2p^2}\Theta^{abc}\Theta^{def}p_a q_b r_c p_d q_e r_f + \frac{\pi}{2p^2}q_l r^l \eta_{cf}\Theta^{abc}\Theta^{def}p_a q_b p_d r_e \;. \end{aligned} \tag{5.89}$$

The t-channel is given by the same expression (5.89) with

$$p = p_3 - p_1, \qquad q = p_3 + p_1, \qquad r = p_4 + p_2 \;, \tag{5.90}$$

and the u-channel is given by (5.89) with:

$$p = p_4 - p_1, \qquad q = p_4 + p_1, \qquad r = p_3 + p_2 \;. \tag{5.91}$$

5.4.2 Large impact parameters

The approximation of large impact parameter corresponds to $t \gg s, u$. In this case the t-channel amplitude dominates. We can Fourier transform to obtain a force that behaves, at large distances, like $\frac{v^2}{r^4}$. The H-propagator indeed generates a force that behaves as $\frac{1}{r^4}$ (because it is a harmonic function only in the transverse directions).

In this approximation with keep only the t-channel and we set:

$$\begin{aligned} p_1 &= (\vec{v} - \frac{1}{2}\vec{p},\ \frac{R_{||}}{2}(\vec{v} - \frac{1}{2}\vec{p})^2,\ \frac{1}{R_{||}}), \\ p_2 &= (-\vec{v} + \frac{1}{2}\vec{p},\ \frac{R_{||}}{2}(\vec{v} - \frac{1}{2}\vec{p})^2,\ \frac{1}{R_{||}}), \\ p_3 &= (\vec{v} + \frac{1}{2}\vec{p},\ \frac{R_{||}}{2}(\vec{v} + \frac{1}{2}\vec{p})^2,\ \frac{1}{R_{||}}), \\ p_4 &= (-\vec{v} - \frac{1}{2}\vec{p},\ \frac{R_{||}}{2}(\vec{v} + \frac{1}{2}\vec{p})^2,\ \frac{1}{R_{||}}). \end{aligned}$$

The notation is:

$$p_i = (\vec{p}_i, p_{i-}, p_{i+})\ , \qquad p_i^2 = \vec{p}_i^{\,2} - 2p_{i-}p_{i+}\ . \tag{5.92}$$

and $R_{||}$ is the radius of the light-like direction of M(atrix)-theory. We set:

$$\begin{aligned} p &= (\vec{p},\ R_{||}(\vec{v}\cdot\vec{p}),\ 0), \\ q &= (2\vec{v},\ R_{||}(\vec{v}^2 + \frac{1}{4}\vec{p}^2),\ \frac{2}{R_{||}}), \\ r &= (-2\vec{v},\ R_{||}(\vec{v}^2 + \frac{1}{4}\vec{p}^2),\ \frac{2}{R_{||}})\ , \end{aligned} \tag{5.93}$$

and assume that $|\vec{p}| \ll |\vec{v}|$.

Following [4, 89, 20], we set the nonzero components of Θ to be $\Theta^{ab+} \equiv \theta^{ab}$, with $a, b = 1 \ldots 4$ and θ is anti-self-dual on $\mathbb{R}^4$. We find that the amplitude is proportional to:

$$A \ \propto\ \frac{32\pi}{R_{||}^2 p^2}\theta^{ab}p_a v_b \theta^{de} p_d v_e + \frac{6\pi}{R_{||}^2 p^2}\vec{v}^2 \eta_{cf}\theta^{ac}\theta^{df}p_a p_d\ . \tag{5.94}$$

After a Fourier transform with respect to $\vec{p}$, this indeed produces a force that is proportional to v^2/r^4.

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