Physics 501 Problem Set 5

Due at the end of class, Friday, March 28^{th} (late assignments will not be accepted).

1. Squeezed states.

In class we defined the translation operator $D(\alpha)$ and the coherent states $|\alpha\rangle$ of the harmonic oscillator with Hamiltonian $H = \hbar\omega(a^\dagger a + 1/2)$,

$$D(\alpha) \equiv \exp(\alpha a^\dagger - \bar{\alpha} a) \ , \qquad |\alpha\rangle \equiv D(\alpha)|0\rangle \ , \qquad a|\alpha\rangle = \alpha|\alpha\rangle \ .$$

Now define the squeezing operator $S(\beta)$ and the squeezed state $|\alpha, \beta\rangle$ with

$$S(\beta) \equiv \exp\left(\frac{\beta}{2}a^2 - \frac{\bar{\beta}}{2}\left(a^\dagger\right)^2\right) \ , \qquad |\alpha, \beta\rangle \equiv D(\alpha)S(\beta)|0\rangle \ .$$

We will use the balanced position and momentum variables, $X = \sqrt{m\omega}x = \sqrt{\hbar/2}(a^\dagger + a)$, and $P = p/\sqrt{m\omega} = \sqrt{\hbar/2}i(a^\dagger - a)$. Recall that the ground state $|0\rangle$ can be represented in the X-P plane as a circle centered at (X,P)=(0,0). The squeezed state can be represented as an ellipse (hence the name, "squeezed"), whose centroid lies at a point (X,P) given by the real and imaginary parts of α and whose eccentricity and orientation are determined by the real and imaginary parts of β. In parts (a-b), you are going to illustrate these facts.

(a) Compute $S^\dagger(\beta)XS(\beta)$ and $S^\dagger(\beta)PS(\beta)$ and us the results to show that

$$S^\dagger(\beta)D^\dagger(\alpha)\begin{bmatrix} X \\ P \end{bmatrix} D(\alpha)S(\beta) \equiv \begin{bmatrix} S^\dagger D^\dagger X D S \\ S^\dagger D^\dagger P D S \end{bmatrix} =$$

$$\begin{bmatrix} \cosh|\beta| - \frac{\mathrm{Re}(\beta)}{|\beta|}\sinh|\beta| & \frac{\mathrm{Im}(\beta)}{|\beta|}\sinh|\beta| \\ \frac{\mathrm{Im}(\beta)}{|\beta|}\sinh|\beta| & \cosh|\beta| + \frac{\mathrm{Re}(\beta)}{|\beta|}\sinh|\beta| \end{bmatrix} \begin{bmatrix} X \\ P \end{bmatrix} + \sqrt{2\hbar}\begin{bmatrix} \mathrm{Re}(\alpha) \\ \mathrm{Im}(\alpha) \end{bmatrix}$$

(b) Describe the effect of the linear transformation in part (a) on a circle centered at zero in the $X - P$ plane (which represents the ground state). Consider two cases: (i) β purely real and (ii) β complex, where it might help to decompose $\beta = |\beta|e^{2i\theta}$.

(c) Show that the root mean squared fluctuations are given by

$$\sqrt{\langle(\Delta X)^2\rangle_{\alpha,\beta}} = \sqrt{\frac{\hbar}{2}}\sqrt{\cosh^2|\beta| + \sinh^2|\beta| - 2\cosh|\beta|\sinh|\beta|\frac{\mathrm{Re}(\beta)}{|\beta|}}$$

and

$$\sqrt{\langle(\Delta P)^2\rangle_{\alpha,\beta}} = \sqrt{\frac{\hbar}{2}}\sqrt{\cosh^2|\beta| + \sinh^2|\beta| + 2\cosh|\beta|\sinh|\beta|\frac{\mathrm{Re}(\beta)}{|\beta|}}$$

Also show that the uncertainty relation is saturated $\sqrt{\langle(\Delta X)^2\rangle_{\alpha,\beta}}\sqrt{\langle(\Delta P)^2\rangle_{\alpha,\beta}} = \frac{\hbar}{2}$ only when β is real. Explain why, using your understanding from part (b).

(d) Show that the time evolution of the squeezed state is given by

$$|\alpha, \beta, t\rangle \equiv e^{-iHt/\hbar}|\alpha, \beta\rangle = |e^{-i\omega t}\alpha, e^{2i\omega t}\beta\rangle \ .$$

Notice, by comparing with part (a), that this is simply a rigid rotation in the $X - P$ plane.

(e) Setting $\hbar = 1$ and $\omega = 1$, and for initial values $\alpha = 10$ and $\beta = 1.5$, plot $\langle X(t)\rangle$, $\langle X(t)\rangle + \sqrt{\langle(\Delta X(t))^2\rangle}$, and $\langle X(t)\rangle - \sqrt{\langle(\Delta X(t))^2\rangle}$, as a function of time, to illustrate the time evolution of X and its uncertainty.

The uncertainty in the electric field presents a fundamental limit for how accurate a distance measurement can be when done with a laser beam. The fact that in the time evolution of a squeezed state there are moments where the uncertainty in E is very small offers an opportunity for more accurate measurements. Notice that a squeezed state is a highly quantum mechanical state, and very different from the ordinary coherent state radiation produced by lasers.

2. A hydrogen atom in the $2p$ state is placed in a cavity. At what temperature of the cavity are the transition probabilities for spontaneous and induced emission equal? Please give an answer in degrees Kelvin.

3. Relativistic limit of the Dirac equation.

The Dirac equation in 4-component spinor notation is, as we saw in class

$$i\hbar\frac{d}{dt}|\psi\rangle = (c\alpha \cdot p + \beta mc^2)|\psi\rangle$$

where α_i and β are 4×4 matrices defined in terms of the Pauli matrices as follows:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad\qquad \beta_i = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

(a) Let's introduce the spin (S) and helicity (h) operators,

$$S_i = \frac{\hbar}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \qquad\qquad h \equiv S \cdot p/|p|$$

Compute the commutator of the the free Dirac Hamiltonian above with the spin S_i and show that it is nonzero, ie, the spin is not conserved on its own.

The helicity is the component of the spin along the direction of the momentum p. Show that helicity commutes with the Hamiltonian, $[h, H] = 0$, and is therefore conserved.

(b) Show that for an electron moving with the speed approaching that of the speed of light, $v \to c$, we can neglect the mc^2 term in the Dirac equation.

(c) Working in the $m = 0$ limit, find simultaneous eigenvectors of the free Dirac Hamiltonian, the momentum p and the helicity h. For each eigenvector, give the values of energy, momentum and helicity.

As we discussed in class, the negative energy solutions correspond to positrons. The two $\pm\hbar/2$ helicity components are usually called the left-handed and the right-handed component.

4. Relativistic atoms.

It was mentioned in class that the problem of the Dirac equation for the hydrogen atom can be solved exactly. The exact energy is

$$E_{nj} = \frac{mc^2}{\sqrt{1 + \frac{\alpha^2}{\left(n-(j+\frac{1}{2})+\sqrt{(j+\frac{1}{2})^2-\alpha^2}\right)^2}}}$$

(a) Expand this energy in powers of the fine structure constant α up to $o(\alpha^4)$. You should obtain the rest mass energy, $o(\alpha^0)$, the non-relativistic energy formula, $o(\alpha^2)$, and the fine structure correction, $o(\alpha^4)$. Please identify each term.

(b) Compute, in Rydbergs, the fine structure contribution to the energy in the S-wave state (l=0, j=1/2) of hydrogen, as a function of n.

(c) Now consider the inner, n=1, shell of a heavy element with atomic number A. Compute the ratio of the $o(\alpha^4)$ term to the $o(\alpha^2)$ term in this case. How large does A have to be before the relativistic effect is 1% of the nonrelativistic energy? Why does the relativistic correction grow with A?