Physics 501 Problem Set 2

Due at the end of class, Wednesday February 6^{th} (late assignments will not be accepted).

1. Entanglement and entropy.

Given a system with a density matrix ρ, define the entropy by

$$S = -\mathrm{tr}\,(\rho \ln \rho)$$

Because $\lim_{\lambda \to 0} \lambda \ln \lambda = 0$, the zero eigenvalues of ρ do not contribute to S.

(a) Show that, for a pure state, the entropy is zero: $S = 0$.

(b) Consider the same set-up of two systems as in question 4 of PS1. I will use the same notation, and you can use the results of that question (PS1 Question 4 part (b) will be particularly useful). We will make the following definitions:

$$S_A = -\mathrm{tr}_A\,(\rho_A \ln \rho_A)$$

$$S_B = -\mathrm{tr}_B\,(\rho_B \ln \rho_B)$$

Show that $S_A = S_B$. The entanglement entropy can be thus measured in either A or B with the same result. The entropy is another measure of the entanglement of the whole system.

(c) For this part of the question, assume that we are studying a system of two spin one half particles, ie that $N = M = 2$, and the bases $\{|a\rangle\}$ and $\{|b\rangle\}$ are just $\{|\uparrow\rangle, |\downarrow\rangle\}$. Show that the EPR state $(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)/\sqrt{2}$ maximizes the entropy in this case, justifying the term 'maximally entangled'.

2. Bell's Theorem for photons.

This question asks you to study the theory behind the Aspect et. al. experiments discussed in class. Check the original papers if anything is unclear.

Consider two photons, entangled in the state:

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\,(H_1 H_2 - V_1 V_2)$$

where H is a horizontally polarized photon, V is a vertically polarized photon, and the subscripts 1 and 2 refer to photons 1 and 2.

(a) Show that this state is invariant under rotations. A rotation by angle θ acts on the H,V basis as follows:

$$H \to (\cos\theta)H + (\sin\theta)V$$

$$V \to (\cos\theta)V - (\sin\theta)H$$

Notice that since the two photons are moving in the opposite directions (back-to-back), rotating the experiment by angle θ about the direction of propagation effectively rotates one photon by θ and the other by $-\theta$.

Alice takes photon 1 and puts it through a polarizer which polarizes the light along direction $\hat{a}$. She defines the result of her experiment, A, to be A=1 if the photon gets through the polarizer and is registered in the photomultiplier behind it. She defines the result of her experiment to be A=-1 if the photon does not get through the polarizer.

Similarly, Bob takes photon 2, and puts it through a polarizer in direction $\hat{b}$. He defines the result of his experiment to be B=1 if his photon gets through the polarizer and B=-1 if the photon does not get through the polarizer.

For each pair of photons, the combined result of the experiment is AB = $\pm$ 1. A large number of experiments is performed, and the average result is denoted by $E(\hat{a}, \hat{b}) \equiv \langle AB \rangle$. Alice and Bob repeat the experiment many times with different polarizer directions, and summarize their result by computing

$$S \equiv E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{b}') + E(\hat{a}', \hat{b}) + E(\hat{a}', \hat{b}')$$

(b) Show that in quantum mechanics the prediction for E is

$$E_{QM}(\hat{a}, \hat{b}) = \cos 2\phi$$

where ϕ is the angle between $\hat{a}$ and $\hat{b}$.

Now let's consider the prediction of a deterministic local hidden variable theory. In such a theory, A is predetermined given a hidden variable λ describing the photon state, and the position of Alice's polarizer, $A = A(\lambda, \hat{a})$. Similarly, $B = B(\lambda, \hat{b})$.

(c) Argue that in the state $|\Phi\rangle$, if Alice and Bob set their polarizers in the same direction, they always measure the same thing (meaning if A=1 then B=1 and if A=-1 then B=-1). Hint: use the rotational invariance you proved in part (a).

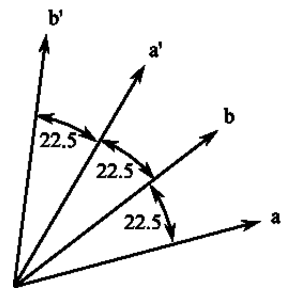
Your result from part (c) can be summarized with $A(\lambda, \hat{a}) = B(\lambda, \hat{a})$. Just like in class, we define, for the hidden variable theory,

$$E(\hat{a}, \hat{b}) = \int d\lambda A(\lambda, \hat{a}) B(\lambda, \hat{b})$$

with $\int d\lambda = 1$.

(d) In the hidden variable theory, prove a Bell's-like inequality for S: $|S| \leq 2$. [If you cannot do this, assume it and do the next part]

(e) Show that if the directions are as shown in the diagram, S = $2\sqrt{2}$ in quantum mechanics, and therefore violates the inequality in part (b).



3. Teleportation.

Consider teleportation with spin one half particles. Alice has a particle, which I will refer to as particle 1, which she wants to teleport to Bob. Her particle is in some state $|\psi\rangle = \alpha|\uparrow\rangle_1 + \beta|\downarrow\rangle_1$. Alice also has particle 2, and Bob has particle 3. 2 and 3 are entangled into the spin zero state. The total wave function of 1, 2 and 3 is therefore

$$(\alpha|\uparrow\rangle_1 + \beta|\downarrow\rangle_1)\,(|\uparrow\rangle_2|\downarrow\rangle_3 - |\downarrow\rangle_2|\uparrow\rangle_3)\,/\sqrt{2}$$

(a) In the four-dimensional Hilbert space of Alice's particles 1 and 2 there are four maximally entangled states. Alice wants to do a single measurement which will collapse the joint state of 1 and 2 onto one of these four states. What observable should she measure? (The answer is a Hermitian operator, and is not unique. Just write down something that works).

(b) With what probability does Alice find each of the four maximally entangled states when she does the measurement?

(c) For each of the possible measurement outcomes, find the state of Bob's particle 3.

(d) For those cases where the state of particle 3 is not identical to $|\psi\rangle$, please describe the physical manipulation Bob must perform on his particle 3 to put it in the state $|\psi\rangle$.

4. Quantum cryptography.

Alice and Bob are trying to establish a code, using quantum cryptography, as explained in class and in the hand-out from Williams and Clearwater. On 3 different occasions, Alice chooses 48 bits (0 or 1), chooses randomly to use the vertical-horizontal or diagonal polarizer for each bit and sends Bob a set of 48 photons. Bob chooses randomly whether to use his vertical-horizontal or diagonal polarizer randomly for each photon and measures the polarizations of the photons. The tables below show the bits chosen by Alice, the polarizers chosen by Alice, the polarizers chosen by Bob and the bits measured by Bob, on the first, second, third and fourth row respectively. On each occasion, Alice and Bob use the first 24 bits sent to test for an eavesdropper. If they conclude that there was probably no eavesdropper then they use the remaining 24 bits to establish the code. In each of these 2 cases do Alice and

Bob determine that there was an eavesdropper? Explain why. If your answer to this question is "no" then give the code that they are able to establish, and compute the probability that they have missed the eavesdropper. (Actually, the probability of missing the eavesdropped is too high for practical purposes, and many more bits than this would need to be sent to test carefully for eavesdropping and to establish a long enough code to be useful, but then it wouldn't fit on a single line!)

(a)

1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
+	×	+	+	×	+	×	×	×	+	×	+	+	+	×	+	×	+	×	×	+	+	×	+
+	×	×	+	×	+	+	+	×	+	×	×	×	+	×	+	+	+	+	×	+	×	+	×
1	0	0	1	0	1	1	0	1	1	0	0	1	0	1	1	0	1	1	0	1	1	0	1

0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0
+	×	×	+	×	+	×	×	+	×	+	+	+	+	×	×	×	+	×	+	+	+	×	×
×	+	×	+	×	+	+	+	×	×	+	+	+	×	+	+	+	+	×	×	+	×	+	+
0	1	1	0	1	0	1	0	0	0	1	0	0	0	1	0	1	0	0	1	1	0	1	1

(b)

1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
+	×	+	+	×	+	×	×	×	+	×	+	+	+	×	+	×	+	×	×	+	+	×	+
+	×	×	+	×	+	+	+	×	+	×	×	×	+	×	+	+	+	+	×	+	×	+	×
1	0	1	1	1	1	1	1	1	1	0	0	0	0	0	1	0	1	1	1	1	0	0	1

0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0
+	×	×	+	×	+	×	×	+	×	+	+	+	+	×	×	×	+	×	+	+	+	×	×
×	+	×	+	×	+	+	+	×	×	+	+	+	×	+	+	+	+	×	×	+	×	+	+
0	0	0	0	1	0	0	1	1	1	1	0	0	0	1	0	0	1	0	1	1	1	0	1

(c) In the hand out, it is claimed that the probability of detecting an eavesdropper using a single bit is 1/4. Please prove this claim, assuming that Eve sets her polarizer at some random angle θ to the directions used by Alice and Bob.