Physics 501 Problem Set 1

Due at the end of class, Wednesday January 23^{th} (late assignments will not be accepted).

General PS policy: you may (and in fact are encouraged to) work in groups and discuss strategies and methods for solving the problems, but you have to write down each solution on your own. If you are stuck, feel free to ask me for hints.

The PS will be graded using the grading rubric attached. Notice that marks are assigned to the quality of your presentation.

1. In class, we have derived that $E_{QM}(\hat{a}, \hat{b}) = -\hat{a} \cdot \hat{b}$. In this question, you will re-derive the same result, using a somewhat more involved, but more canonical, approach.

Consider 2 spin one-half particles. Their total Hilbert space is four dimensional and is spanned by $|\uparrow\rangle|\uparrow\rangle$, $|\uparrow\rangle|\downarrow\rangle$, $|\downarrow\rangle|\uparrow\rangle$, and $|\downarrow\rangle|\downarrow\rangle$. In matrix notation, let's denote this basis with

$$|\uparrow\rangle|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |\uparrow\rangle|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |\downarrow\rangle|\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |\downarrow\rangle|\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(a) In this notation, what does the EPR pair $|EPR\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$ look like?

(b) When Alice sets her Stern-Gerlach apparatus to point in the $\hat{a}$ direction, and Bob sets his in the $\hat{b}$ direction, the combined observable is $K \equiv (2\hat{a} \cdot J) \otimes (2\hat{b} \cdot J)$ where Alice's (the first) component acts on the first particle and Bob's (the second) component acts on the second particle. The Js are just the Pauli matrices,

$$J_x = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad J_y = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad J_z = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

What is K in the matrix notation?

(c) Show that $E_{QM}(\hat{a}, \hat{b}) \equiv \langle EPR|K|EPR\rangle = -\hat{a} \cdot \hat{b}$.

2. Before you attempt this question, please read the article by Mermin 'Is the moon there when nobody looks? Reality and the quantum theory' (Physics Today, vol 38, pg 38, (April 1985)). It might help you appreciate the power of Bell's theorem. In this question we will generalize the Mermin Machine.

The Generalized Mermin Machine (GMM) differs from the one in the article in only one way: it has more switch positions. It works as follows: there is an emitter and two detectors.

These three parts do not talk to each other, except via the particles which the emitter emits towards the two detectors. Each detector is equipped with a switch which can be set to one of N positions. Each detector has on top two light bulbs, a green one and a red one. When the emitter's button is pressed, it emits particles towards the detectors, which then light up one of their light bulbs. Running this experiment with random switch settings we observe the following:

- If the switches are set the same, the light bulbs light up either both red or both green.
- If we average over all the measurements ignoring the switches, half the time the light bulbs have the same colour and half the time different colours.

Can this be a result of local, deterministic physics?

Mermin in his article provides a proof that it cannot for $N = 3$. (HINT: Writing out a table describing all possibilities should help to understand his proof and to do this question.)

(a) Show that for $N = 4$, the GMM can be made to work with purely classical physics (ie: construct a local, classical model for the $N = 4$ GMM).

(b) Show that for $N = 5$, there is no classical way of constructing the GMM, just like there was no way to do it for $N = 3$. (HINT, the important thing here is that $N = 5$ is odd).

3. Alice and Bob each have a detector which measures the signal sent from a black-box central emitter. Both Alice's and Bob's detectors are equipped with switches, which can be set to one of two positions, labeled I and II. The detectors also have a green and a red light-bulb each. When the 'go' button on the emitter is pressed, the emitter sends signals to both detectors simultaneously, and one of the two light-bulbs on each detector lights up. Which light-bulb lights up on each detector depends on the setting of the detector's switch, and the signal sent to it. The detectors are not in any direct contact with each other.

Alice and Bob make many, many measurements and observe the following rules:

1. If both switches are set to I, the light-bulbs are never green at the same time.
2. If Alice's switch is set to I and Bob's is set to II, then if Alice gets a red light-bulb, then Bob always gets a green light-bulb.
3. If Bob's switch is set to I and Alice's is set to II, then if Bob gets a red light-bulb, then Alice always gets a green light-bulb.

(a) Assuming classical local realism, show that when both switches are set to II, Alice and Bob cannot simultaneously get red light-bulbs.

Now consider the quantum version of this apparatus. The emitter emits a pair of entangled spin-half particles, in the total quantum state given by

$$\sqrt{1-2x}|\uparrow\rangle_A|\uparrow\rangle_B + \sqrt{x}|\downarrow\rangle_A|\uparrow\rangle_B + \sqrt{x}|\uparrow\rangle_A|\downarrow\rangle_B$$

where x is some fixed real number between 0 and 1/2, known to both Alice and Bob.

When the switch is in the I position, the detectors measure the spin of the particles in the $\uparrow, \downarrow$ basis. If the measurement comes out $\uparrow$, the detector lights up the red light-bulb, when it comes out $\downarrow$, the detector lights up the green arrow.

(b) Show that this arrangement guarantees that rule 1 above is satisfied.

(c) Figure out the bases in which the spins must be measured when the switch is in the II position to satisfy rules 2 and 3. The basis is allowed to depend on x.

(d) When both switches are set to II, compute the probability that Alice and Bob simultaneously get red light-bulbs, as a function of x. This shows that in Quantum Mechanics there is a finite violation of the classical result in part (a).

4. This question is basically an exercise in linear algebra. It requires careful mathematical treatement. The goal is to construct a measure of entanglement known as the Schmidt decomposition.

In class, we considered a $N \times M$ matrix C_{ab} corresponding to a pure state $|\psi\rangle$ in a tensor product of two Hilbert spaces H_A and H_B with dimensions N and M respectively. The orthonormal bases for the two Hilbert spaces are $\{|a\rangle\}$ and $\{|b\rangle\}$ with $a = 1..N$ and $b = 1..M$. The connection between C and $|\psi\rangle$ is

$$|\psi\rangle = \sum_{ab} C_{ab}|a\rangle|b\rangle$$

The density matrix corresponding to tracing over H_B is

$$\rho_A = \sum_{aa'} (CC^\dagger)_{aa'}|a'\rangle\langle a|$$

and that corresponding to tracing over H_A is

$$\rho_B = \sum_{bb'} (C^\dagger C)_{bb'}|b\rangle\langle b'|$$

(a) Let $CC^\dagger$ have $n \leq N$ positive eigenvalues (recall than in general, the eigenvalues of $CC^\dagger$ are nonnegative). Denote by v^i, $i = 1..n$ the eigenvectors of $CC^\dagger$ having nonzero eigenvalues λ^i, ie $CC^\dagger v^i = \lambda^i v^i$. Prove that it is possible to chose those v^is so that they form an orthonormal set.

(b) Now define $u^i = C^\dagger v^i/\sqrt{\lambda^i}$. Prove that the u^is are eigenvectors of $C^\dagger C$ and are orthonormal. What are the nonzero eigenvalues of $C^\dagger C$?

(c) Define states in H_A and H_B respectively by $|i\rangle_A \equiv \sum_a v_a^i |a\rangle$ and $|i\rangle_B \equiv \sum_b \bar{u}_b^i |b\rangle$. Prove that the $|i\rangle_A$s and the $|i\rangle_B$s are orthonormal sets, and show that $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_i \sqrt{\lambda_i} |i\rangle_A |i\rangle_B$$

[Hint: Prove that if $C^\dagger C u = 0$ for some u then it follows that $Cu = 0$. Use this to enlarge the set of u^is to get a complete basis for H_B]

This way of writing $|\psi\rangle$ as a diagonal sum over two orthonormal sets is known as the Schmidt decomposition. The number of non-zero eigenvalues n is known as the Schmidt number (it is the same as the rank of C).

(d) Show that if $n = 1$ then both ρ_A and ρ_B are pure states. Also show that if $n > 1$ then they are not. The Schmidt number is thus a measure of the entanglement of the system.