Physics 313 Problem Set 9

Important concepts from lectures 28,29,30

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Maxwell speed distribution $\mathcal{D}(v)$. This is a distribution function - meaning that when you integrate over it, you get probabilities.

$$\text{probability}(v_1 < v < v_2) = \int_{v_1}^{v_2} \mathcal{D}(v)\ dv$$

$$\mathcal{D}(v) \quad = \quad \left(\frac{m}{2\pi kT}\right)^{3/2} \ 4\pi v^2 \ e^{-mv^2/2kT}$$

The first factor comes from normalization ($\int \mathcal{D}(v)dv = 1$) the second is the area of the velocity sphere and the third is the Boltzmann factor.

The distribution peaks at v_{max}, the most likely speed

$$v_{max} = \sqrt{2\frac{kT}{m}}$$

Using the Maxwell distribution, we can calculate mean values for any quantity X as

$$\overline{X} \equiv \int \mathcal{D}(v)\ X\ dv$$

To evaluate the integral, you will need to express X as a functon of the speed v.

For example the mean velocity

$$\overline{v} \equiv \int \mathcal{D}(v)\ v\ dv = \sqrt{\frac{8}{\pi}\frac{kT}{m}}$$

and the root-mean-square velocity

$$v_{RMS} \equiv \sqrt{\int \mathcal{D}(v)\ v^2\ dv} = \sqrt{3\frac{kT}{m}}$$

Notice that all these velocities grow with (the square root of) temperature.

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$$F = -kT \ln Z$$

From this formula, you can derive S, P and μ (all as functions of T, V and N) by taking the appropriate partial derivatives, for example

$$S = -\frac{\partial F}{\partial T}|_{V,N} = k \ln Z - \frac{U}{T}$$

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For a system with multiple component, say n of them, each with its own partition function Z_i, the total partition function is just the product of the individual ones:

$$Z_{tot} = Z_1 Z_2 \ldots Z_N$$

Therefore, if you have N noninteracting particles, each with a 1-particle partition function Z_1, the total partition function is

$$Z = (Z_1)^N \quad \text{for distinquishable particles}$$

$$Z = \frac{1}{N!}(Z_1)^N \quad \text{for indistinquishable particles}$$

This last formula is only true in the dilute gas regime (where the number of particles is much, much less than the number of accessible states). If this does not apply, we enter the regime of quantum gas, and have to use the grand partition function instead (the next subject in the course).

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We have rederived the formula for the entropy of the ideal gas, no longer restricting ourselves to the monatomic case, and obtained

$$S(T, V, N) = kN \left[\ln\left(\frac{V}{N}\right) + \ln\left((kT)^{f/2}\right) + \frac{f+2}{2} + \text{constant} \right]$$

where the constant depends on the mass and moment of inertia of the molecule as well as the Planck constant h.

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Problem Set

Due at the end of class, Friday November 21^{st} (late assignments will not be accepted).

1. Schroeder Problem 6.31, pg 240.

2. Schroeder Problem 6.39, pg 246.

3. The kinetic energy of a single molecule in the ideal gas is of course $E = mv^2/2$.

(a) Using the Maxwell distribution, compute the average kinetic energy, $\overline{E}$. Does it agree with the equipartition theorem?

(b) Compute the average of the kinetic energy squared $\overline{E^2}$.

(c) Put these two things together and find the root-mean-squared fluctuation in the kinetic energy $(\Delta E)_{RMS} \equiv \sqrt{\overline{E^2} - \left(\overline{E}\right)^2}$

You might find the following integrals useful in this question, or in other questions having to do with the Maxwell distribution

$$\int_0^\infty x^2 \, e^{-x^2} \, dx = \frac{\sqrt{\pi}}{4}$$

$$\int_0^\infty x^3 e^{-x^2} \, dx = \frac{1}{2}$$

$$\int_0^\infty x^4 \, e^{-x^2} \, dx = \frac{3\sqrt{\pi}}{8}$$

$$\int_0^\infty x^5 e^{-x^2} \, dx = 1$$

$$\int_0^\infty x^6 \, e^{-x^2} \, dx = \frac{15\sqrt{\pi}}{16}$$

In case you are curious, the general formula (in terms of the gamma function, explained on page 387 in the book) is

$$\int_0^\infty x^n \, e^{-x^2} \, dx = \frac{1}{2} \, \Gamma\left(\frac{n+1}{2}\right)$$

This is true even if n is not an integer.

4. In this question, you will analize the spin-1 paramagnet, which is similar to the spin-1/2 paramagnet. Each spin has one of three available states, which we will label by $j = -1, 0, 1$. The magnetic moment of a spin in state j is μj (where μ is the amplitude of the magnetic moment of the spin, some known constant). In a magnetic field, the energy in state j is $E(j) = -(\mu B)j$.

(a) Find the average magnetic moment and the average energy of a single spin at temperature T (as a function of mu, B and T).

(b) Plot the total magnetization of N spins (assuming they don't interact with each other) as function of β and compare with figure 3.11 in Schroeder.

(c) Show that the Curie law (magnetization is inversly proportional to temperature, for large T) holds.

(d) Find the free energy of N such spins.

(e) Find the entropy as a function of temperature and plot it. (Notice that you have done this before for the spin 1/2-paramagnet in question 3 of Problem Set 5, using less convenient methods.)

5. Using the formula for entropy of the ideal gas show that for quasistatic, adiabatic expansion, $VT^{f/2}$ is a constant (Do not derive it the way we did in class long time ago, start with the entropy instead.)

6. Schroeder 7.10, page 265.

Extra: Schroeder Problem 6.41 is a great exercise if you want to make sure you understand all the math in the Maxwell distribution.