Physics 313 Problem Set 10

Important concepts from lectures 32-35

To deal with (noninteracting) quantum gases we changed our focus from particles to 1-particle states and allowed the number of particles N to fluctuate, making the chemical potential μ constant instead. This is analogous to fixing the temperature and allowing the internal energy to fluctuate, which lead us to the Boltzmann factor. Fixing μ and allowing N to fluctuate leads to the Gibbs factor, the probability of having n particles is state s which has energy E_s

$$P(s, n_s) \sim e^{-\beta(E_s-\mu)n_s}$$

The grand partition function for a single particle state with energy ϵ is

$$\mathcal{Z}(T, \mu, \epsilon) \equiv \sum_n e^{-\beta(\epsilon-\mu)n} \tag{1}$$

where the sum is over **all allowed particle numbers** in that state.

To obtain the total grand partition function for the whole system, multiply the individual $\mathcal{Z}$ together for all states

$$\mathcal{Z}(T, \mu, V) \equiv \prod_s \mathcal{Z}(T, \mu, \epsilon(s))$$

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The grand partition function is typically used to calculate the average occupation number, and all further calculations start from there. There are two cases:

- **Bosons** - any number of bosons can occupy the same state (Bose-Einstein statistics). The sum in (1) is therefore over all n from 0 to infinity. We have shown that

  $$\mathcal{Z} = \frac{1}{1 - e^{-\beta(\epsilon-\mu)}}$$

  and that the average occupation number is

  $$\overline{n}_{BE} = \frac{1}{e^{\beta(\epsilon-\mu)} - 1}$$

- **Fermions** - two or more fermions can never occupy the same state (Pauli exclusion principle). This gives the Fermi-Dirac statistics. The sum in (1) is therefore reduced to $n = 0, 1$. We have then that

  $$\mathcal{Z} = 1 + e^{-\beta(\epsilon-\mu)}$$

  and the average occupation number is

  $$\overline{n}_{FD} = \frac{1}{e^{\beta(\epsilon-\mu)} + 1}$$

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Notice that the formulas for bosons make no sense if $\mu > \epsilon$. When μ is raised as high as the lowest energy state in any system, all the particles collapse to that one state! this is called Bose-Einstein condensation (BEC) – check out the course website for links to more info on this topic.

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One of the most important examples of Bose-Einstein statistics is Black Body Radiation (BBR).

The energy spectrum of a BBR in a box is the so-called Planck distribution

$$\mathcal{I}(\epsilon) = \frac{8\pi \left(\frac{\epsilon}{hc}\right)^3}{e^{\beta\epsilon} - 1}$$

Just like the Maxwell speed distribution, this function lives to be integrated. The most intense energy (the peak of $\mathcal{I}(\epsilon)$) is at $\epsilon_{MAX} = (2.82)kT$ (Wien's law). The total energy per volume is

$$\frac{U}{V} = \int \mathcal{I}(\epsilon) d\epsilon = \frac{8\pi^5}{15} \frac{(kT)^4}{(hc)^3} \sim T^4$$

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One of Nature's nearly perfect examples of BBR is the Cosmic Microwave Background (check out the course website for more info on this topic).

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A Black Body (BB) is something whose surface absorbes all indident light. Power radiated by the surface of a BB per its area is (Stefan's law)

$$\frac{\text{power}}{A} = \frac{c}{4} \frac{U}{V} = \sigma T^4$$

where σ is the Stefan-Boltzmann constant

$$\sigma = \frac{2\pi^5}{15} \frac{k^4}{(hc)^3} = 5.67 \cdot 10^{-8} \frac{W}{m^2 K^4}$$

A BB radiates with the Plank spectum above, propertly normalised.

Problem Set

Due at the end of class, Friday November 28^{th} (late assignments will not be accepted).

1. (a) Show that the peak of the Planck spectrum as a function of energy is at $\epsilon_{MAX} = 2.82kT$.

(b) Rewrite formula (7.83) in the book so that the integration is over the wavelength $\lambda = hc/\epsilon$ instead over the energy ϵ and find λ_{MAX}, where the wavelength distribution peaks. Note: you will **not** get that $\lambda_{MAX} = hc/\epsilon_{MAX}$.

(c) Find λ_{MAX} for the Sun (T=5800K). What color does that correspond to? (look it up!)

2. Show that the photon density in a BBR gas of photons is proportional to T^3 (you do not need to calculate the coefficient of this proportionality).

3. Schroeder 7.45, page 297.

4. Schroeder problem 7.55, page 307.

5. Schroeder problem 7.56, page 307.

6. A satelite is in orbit around the Earth, where the intensity of solar radiation (the so called solar constant) is 1370 W/m^2. The surface temperature of the Sun is 5800K.

(a) Imagine the temperature of the Sun suddenly decreased by a factor of 2. What would then be the intensity of radiation experienced by the satellite?

(b) The satelite has an array of solar batteries, pointed at the Sun., absorbing the blackbody radiation the Sun is emitting. The total area of this array is $5m^2$. To be converted into electricity, the energy of the absorbed photon must be at least 1.5eV. Any photon with energy below this threshold is wasted. Any photon with energy above the threshold is converted to electrical power with 80% efficiency. Estimate the electrical power (in watts) the array is producing.

You will find the following table of integral values useful:

y	$\int_0^y \frac{x^3}{e^x-1}dx$
1	0.225
2	1.18
3	2.55
4	3.88
5	4.90
6	5.59
∞	$\pi^4/15 = 6.49$