+ **1.** To get a better idea of how the binomial coefficients behave for large numbers, we need to obtain something called the Stirling Approximation. Start with

$$\ln N! = \sum_{i=1}^{N} \ln i \approx \int_0^N dx \, \ln x$$

and show that $\ln N! \approx N \ln N - N$. This is the lowest order Stirling approximation.

$$\ln N! \simeq \int_0^N dx \quad \ln x \quad = x \ln x - x \Big|_0^N = N \ln N - N$$

SINCE $\lim_{x \to 0} x \ln x = 0$

+ **2.** Consider now $\ln \binom{N}{\alpha N}$ where α is between 0 and 1. Use the Stirling approximation to approximate this quantity when N is large.

$$\ln \binom{N}{\alpha N} = \ln N! - \ln(\alpha N)! - \ln((1-\alpha)N)!$$

$$\simeq N \ln N \cancel{- N} - \alpha N \ln(\alpha N) + \cancel{\alpha N} - (1-\alpha) N \ln((1-\alpha)N) + \cancel{(1-\alpha)N}$$

$$= N\left(\cancel{\ln N} - \alpha \ln \alpha - \cancel{\alpha \ln N} - (1-\alpha)\ln(1-\alpha) - \cancel{(1-\alpha)\ln N}\right)$$

$$= -N\left(\alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)\right)$$

+ **3.** Show that $\binom{N}{\alpha N}$ has a maximum for $\alpha = \frac{1}{2}$.

FOR $\alpha = 0$ AND $\alpha = 1$, $\binom{N}{\alpha N} = 1$

BY SYMMETRY $(\alpha \to 1-\alpha)$, EXTREMUM AT $\alpha = 1/2 \Rightarrow$ DERIVATIVE VANISHES

SINCE $\binom{N}{\frac{1}{2}N} > 1$, IT'S PROBABLY A MAXIMUM.

+ **4.** Use the Taylor approximation to write $\ln \binom{N}{(\frac{1}{2}+x)N} \approx \ln \binom{N}{\frac{1}{2}N} + ax^2$ for small x and find a. Using this, show that the plots on the previous page are approximately gaussian and that their width is proportional to $1/\sqrt{N}$.

$$\ln \binom{N}{(\frac{1}{2}+x)N} \simeq -N\left(\left(\tfrac{1}{2}+x\right)\ln\left(\tfrac{1}{2}+x\right) + \left(\tfrac{1}{2}-x\right)\ln\left(\tfrac{1}{2}-x\right)\right)$$

$$= +N \ln 2 - N\left(\left(\tfrac{1}{2}+x\right)\ln(1+2x) + \left(\tfrac{1}{2}-x\right)\ln(1-2x)\right)$$

$$\simeq N \ln 2 - N\left(\left(\tfrac{1}{2}+x\right)(2x - 2x^2) + \left(\tfrac{1}{2}-x\right)(-2x - 2x^2)\right)$$

$$= N \ln 2 - 2Nx^2 \qquad a = -2N$$

$$\therefore \binom{N}{(\frac{1}{2}+x)N} = e^{N \ln 2 - 2Nx^2} = 2^N e^{-2Nx^2}$$

GAUSSIAN, WIDTH $\sim \frac{1}{\sqrt{N}}$