

Tutorial #11: Wavepackets

In order to explain how an electron can “go through both slits” in a double slit experiment, we introduced the idea that a single electron can be in a state that is a quantum superposition of electron states with definite positions (the POSITION EIGENSTATES). For example, if we have a state

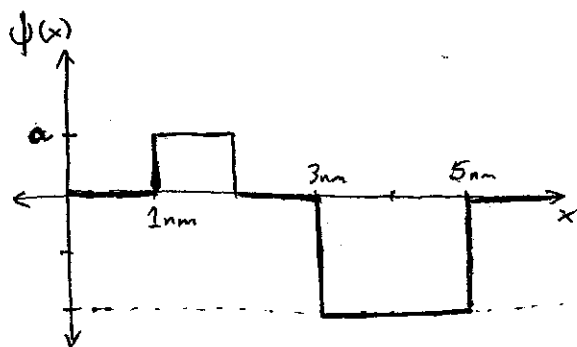
$$\frac{1}{\sqrt{2}} |x_1\rangle + \frac{1}{\sqrt{2}} |x_2\rangle$$

the electron does not have a definite location before we measure it, but if we measure the position, we will either find x_1 or x_2 , each with probability $1/2 = (1/\sqrt{2})^2$.

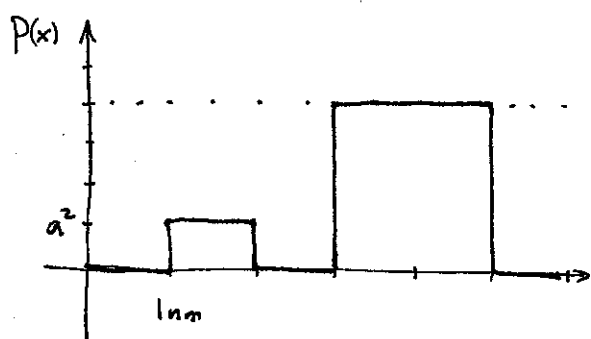
In reality, electron states will never be superpositions of just a few position eigenstates. Instead, they will be combinations of an infinite number of them, with the superposition described by a wavefunction $\psi(x)$ that tells us the relative amount of the $|x\rangle$ eigenstate in the superposition. In this case, the probability of measuring the electron at some exact location x will be zero (since there are an infinite number of possible locations), but $|\psi(x)|^2$ tells us the probability density of finding the electron near x .

The x in the wavefunction usually represents coordinates (x,y,z) (i.e. the wavefunction is a function of three-dimensional space), but we will often discuss situations, such as a thin wire, where the electron is only allowed to move in one direction. In this case, $\psi(x)$ is just a function of one variable.

Question 1



a) The plot at the left shows the wavefunction for an electron in a thin wire. On another graph, sketch the probability density for finding the electron. ★ Have $P(x) = |\psi(x)|^2$ ★



We want $\int_{-\infty}^{\infty} dx P(x) = 1$ so:

$$9 \text{ nm} \cdot a^2 = 1$$

$$\Rightarrow a = \frac{1}{3 \text{ nm}^{1/2}}$$

b) For this wavefunction, what is the probability that we will find the electron between 1 nm and 2 nm in a measurement? What is the probability that we'll find it between 3 nm and 5 nm ? (Hint: how can we figure out the value of a ?)

$$P_{1 < x < 2} = \int_{1 \text{ nm}}^{2 \text{ nm}} dx P(x) = \int_{1 \text{ nm}}^{2 \text{ nm}} dx \cdot \frac{1}{9 \text{ nm}} = \frac{1}{9}$$

or: area of big box is 8 x area of small box \therefore

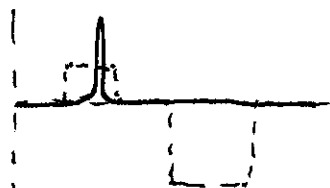
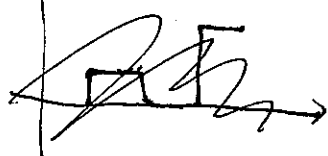
$$P_{3 < x < 5} = \int_{3 \text{ nm}}^{5 \text{ nm}} dx P(x) = \int_{3 \text{ nm}}^{5 \text{ nm}} dx \frac{4}{9 \text{ nm}} = \frac{8}{9}$$

$$P_{1 < x < 2} = \frac{1}{8} P_{3 < x < 5}$$

$$\text{But } P_{1 < x < 2} + P_{3 < x < 5} = 1$$

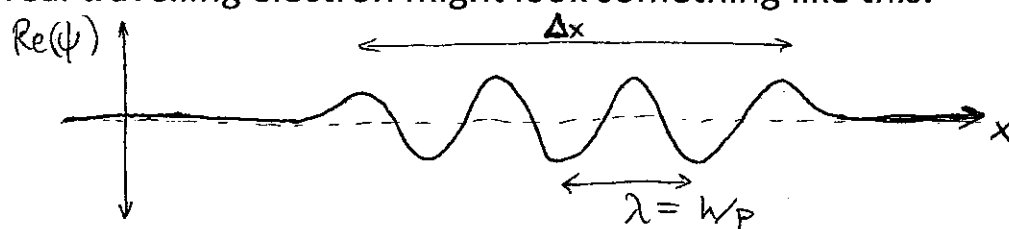
$$\text{so } P_{1 < x < 2} = \frac{1}{9}$$

c) On the original diagram, sketch a possible wavefunction for the electron immediately after we make a measurement of position.



Question 2

The wavefunction for an electron is allowed to be any complex function such that the integral of $|\psi|^2$ is equal to 1. But in order to explain the diffraction patterns that are observed with electrons having a definite momentum, we expect that the wavefunction for a travelling electron with momentum p should look and behave like a wave with wavelength $\lambda = h/p$. As we discussed in class, real-world waves always have finite extent (and our intuition tells us that the wavefunction for a real electron wouldn't be infinitely spread out). Thus, the wavefunction for a real travelling electron might look something like this:



We call this shape a WAVEPACKET, and we can see that it is characterized by a wavelength λ and a spread Δx which can be varied independently.

In this question, we will discover that wavepackets are actually built from a superposition (i.e. a sum) of mathematical pure waves (like $\cos(2\pi x/\lambda)$) with a range of wavelengths near λ . This range becomes larger and larger for smaller Δx , so the narrower the wavepacket, the less precisely we can say what the wavelength (or in turn the momentum is). In other words, *the more accurately that we know the position of an electron, the less accurately we know its momentum.*

To begin, open up the Fourier: Making Waves simulation, which may be found here:

http://phet.colorado.edu/simulations/sims.php?sim=Fourier_Making_Waves

a) The top panel has some parameters A_1, A_2 , etc... that can be varied either by typing in a value or dragging the coloured bars up and down. What is the function displayed in the second and third panel when A_k is nonzero and all but one of the A s are zero (write a precise formula in terms of k and A_k)? *Hint: first do it for A_1 , then A_2 , etc...until you can guess the general formula.*

$$A_k \sin(2\pi k x)$$

b) For your function in a), what is the wavelength (in terms of k)?

$$\lambda = \frac{1}{k}$$

c) When more than one of the A s is non-zero, the second panel shows the different functions separately, while the third panel adds up all the contributions. You might want to use the +/- buttons to adjust the vertical scale. Leave the horizontal axis to go from -0.5 to 0.5 for now. Try setting $A_5 = 0.5$, $A_6 = 1$, and $A_7 = 0.5$. What is a mathematical equation for the function shown in the third panel?

$$0.5 \sin(10\pi x) + \sin(12\pi x) + 0.5 \sin(14\pi x)$$

d) Hopefully, this looks something like the wavepacket that we drew above. What are λ and Δx in this case? For the contributing waves, what range of values does $1/\lambda$ take?

$$\lambda \approx \frac{1}{6}$$

$$\Delta x = 0.6 - 0.8$$

$$\cancel{A(x)} \quad 5 \leq \frac{1}{\lambda} \leq 7$$

e) Now let's try building a wavepacket with a broader range of wavelengths. In the tab at the top right (just under "Preset Functions"), pick "wavepacket" (and adjust the vertical scale so it doesn't go out of the box). What range of values for $1/\lambda$ are in the superposition now? What are λ and Δx for the new wavepacket?

$$1 \leq \frac{1}{\lambda} \leq 11$$

$$\lambda \approx \frac{1}{6}$$

$$\Delta x \approx 0.25$$

f) Qualitatively, how does the new wavepacket with a broader range of wavelengths (but the same central wavelength) differ from the wavepacket of parts c) and d)?

It is narrower but has approx. the same wavelength.

Your observations point to a mathematical fact about adding up pure waves. In order to get a packet that appears more localized, we need to add waves with a broader range of wavelengths. This has a dramatic physical interpretation. The pure waves have a single definite wavelength, and thus represent the wavefunctions for particles with a single definite momentum. These are **MOMENTUM EIGENSTATES**. Since a wavepacket is a superposition of these, it represents a particle without a definite momentum (i.e. there is some *uncertainty* in momentum). But we know that such particle also has some uncertainty in position, determined by how spread out the wavepacket is. So physically, our conclusion is that *in order to decrease the uncertainty in position, we need to increase the uncertainty in momentum*. This can be turned into a precise physical law known as the **HEISENBERG UNCERTAINTY PRINCIPLE**.

Question 3

The functions that we built in the previous question are actually not true wavepackets. To see why, start again by choosing "wavepacket" from the "Function:" tab in the "Preset Functions" box. Now expand the horizontal scale to go from -2 to 2. You should see that the wavepacket shape actually repeats itself. In a real wavepacket, there should be only one "packet," so we need to figure out how to get rid of the others.

a) What is the distance D between the wavepackets here?

$$D \approx 1$$

b) Starting from the preset wavepacket, what happens if you set all the odd A 's (A_1, A_3, \dots) to zero? What are D , λ , and Δx now?

$$D \approx 0.5$$

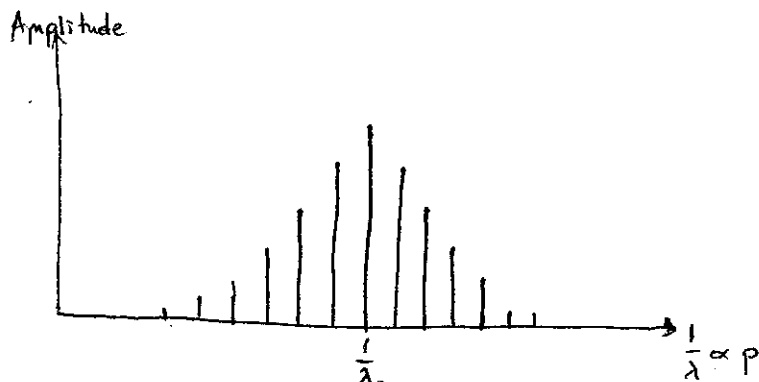
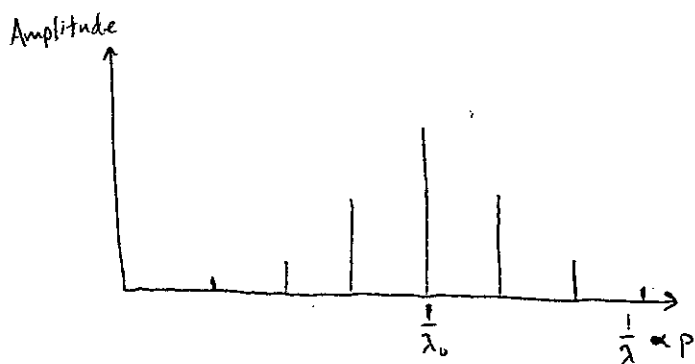
$$\lambda \approx \frac{1}{6}$$

$$\Delta x \approx 0.25$$

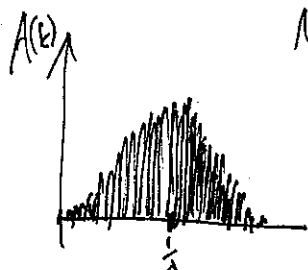
} about the same as before

c) The sketch below shows the amplitudes for waves of various inverse wavelengths (proportional to momenta) which sum up to a set of repeated wavepackets. Based on your observations in part b), sketch a similar graph that would represent a set of waves adding up to repeated wavepackets with the same λ , and Δx , but twice the distance D between the packets.

want $D \rightarrow 2D$
with $\lambda, \Delta x$ fixed



d) If we wanted the distance between the wavepackets to go to infinity (so that we really just have one single wavepacket), what superposition of pure waves would we need? How can you represent that in a diagram similar to the ones in part c?



Need to include same range of wavelengths, but infinite number of different wavelengths in this range.

To check your answer, click on the "Discrete to Continuous" tab at the top of the simulation. The three panels have the same interpretation as before, but now we can add more wavelengths. Expand the horizontal scale as far as it will go (from -8 to 8). Describe what happens in the top and bottom panels as you move the slider in the top right box. Does this agree with what you expected in d?

slide right
top picture: ~~more~~ less wavelengths

slide left
more wavelengths (same range)

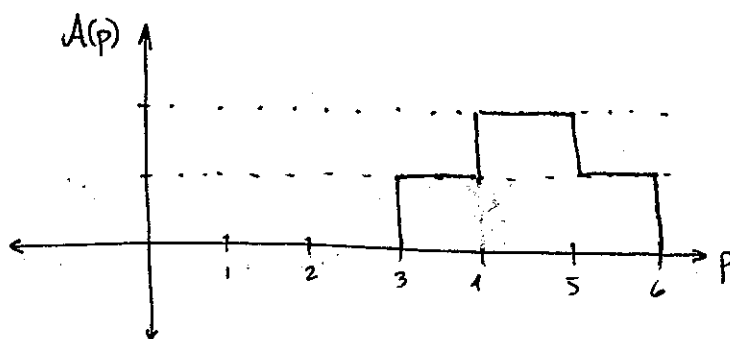
bottom picture: packets closer together

packets further apart.

Question 4

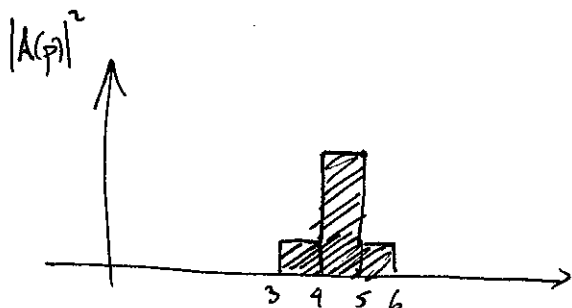
We have now seen that true wavepackets are obtained by adding up pure waves with a continuous range of wavelengths around some central value. For this question, we'll focus on the true wavepackets, so set the "Spacing..." slider all the way to the left.

First, let's review the physical interpretation of these two pictures. The bottom picture is the wavefunction for a travelling electron. It shows what superposition of position eigenstates this electron state is made from. The top picture shows how to build the same wavepacket by using a superposition of momentum eigenstates (i.e. pure waves). Thus, it is just like a wavefunction for momentum.



a) Suppose an electron wavepacket is built from a superposition of momentum eigenstates described by the graph above. If we make a measurement of momentum, what is the probability that the momentum will be between 5 and 6 (in the units used for the graph)? *Hint: the same rules that applied to our quantum superpositions of position eigenstates apply to quantum superpositions of momentum eigenstates.*

$$\text{Relative Probability} \sim \left| \text{coeff in quantum superposition} \right|^2 = |A(p)|^2$$



Between 5 & 6 : $\frac{1}{6}$ of area

$$\therefore P_{5 < p < 6} = \frac{1}{6}$$

b) Now that we understand the physical implications of the top and bottom pictures (the momentum and position wavefunctions), it should be clear that the more spread out the position wavefunction is, the more uncertainty there will be in a measurement of position. Similarly, the more spread out the momentum wavefunction (top picture) is, the more uncertainty there will be in a measurement of momentum.

We saw earlier that to decrease position uncertainty, we need to increase momentum uncertainty. To make this more precise, compare the widths of the top and bottom pictures when the top "Wave packet width" slider is set to π vs 2π :

don't worry about exactly what σ_k means. \rightarrow

	Spread in top graph Δk	Spread in bottom graph Δx
$\sigma_k = \pi$	$\sim 4\pi$	~ 1
$\sigma_k = 2\pi$	$\sim 8\pi$	$\sim \frac{1}{2}$

To reduce the position uncertainty (spread in bottom picture) by half, what do we have to do to the momentum uncertainty?

Double it

c) Finally, let's see what happens when we change the "Wave packet center" slider.

As you move the slider, look at the top picture. How do the central value of momentum and the range Δp of momenta in the superposition change as you move the slider (recall that $k=2\pi/\lambda$ is proportional to p)?

Δp stays same

central value changes

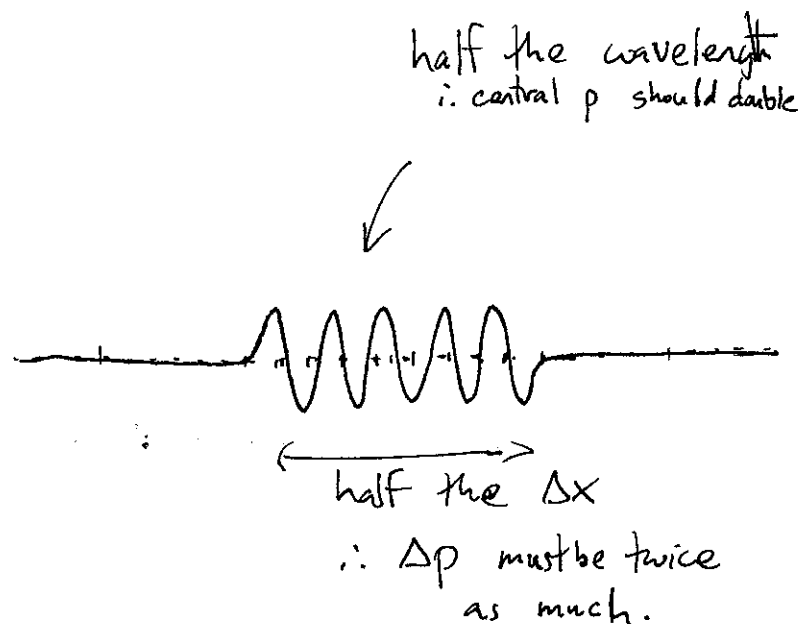
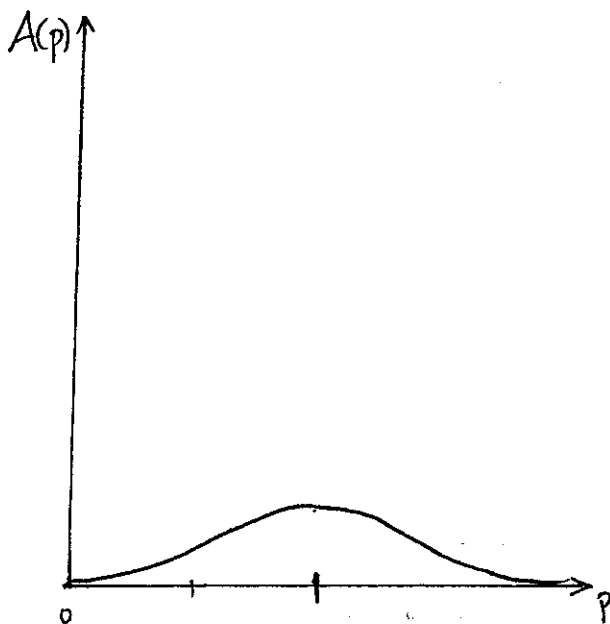
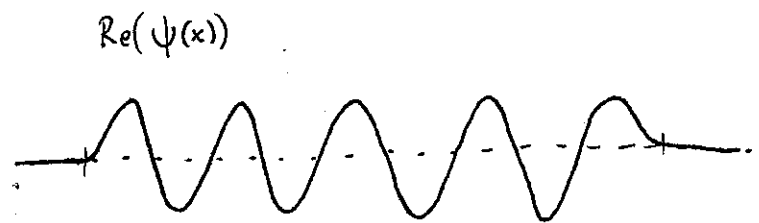
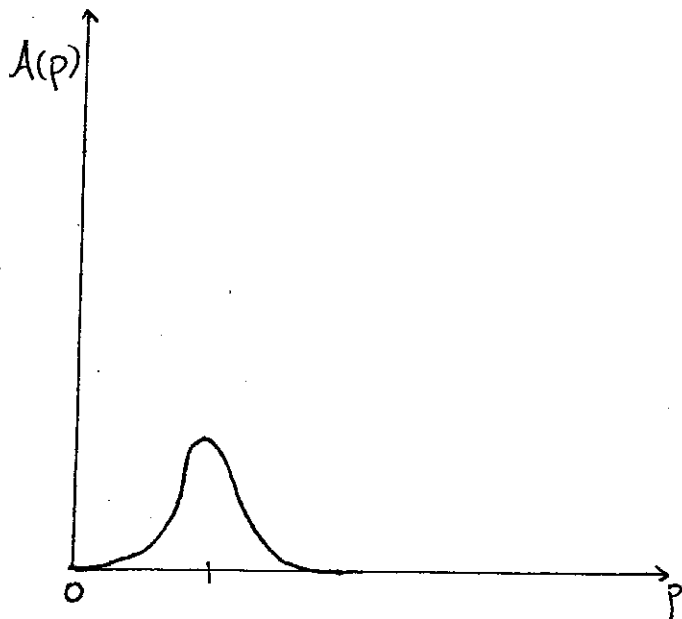
How do λ and Δx for the position wavefunction in the bottom picture change?

Δx stays same

λ increases when p decreases

Question 5

Tying everything together now, suppose the graph below describes the superposition of momentum eigenstates for the electron wavepacket shown on the right. Fill in the bottom graph to show what combination of momentum eigenstates we need to make the wavepacket shown at the bottom right.



draw your graph here.