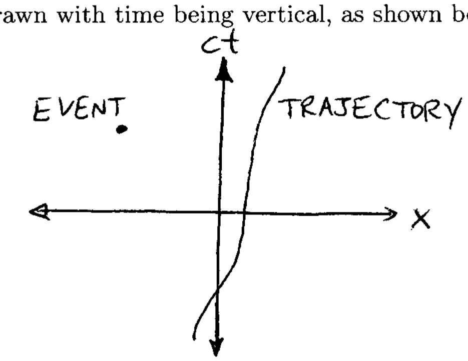
Notes on Spacetime Diagrams

Spacetime diagrams are a useful tool for visualizing events and trajectories and especially for understanding the Lorentz transformations that give us the coordinates and times for an observer moving relative to the original frame of reference.

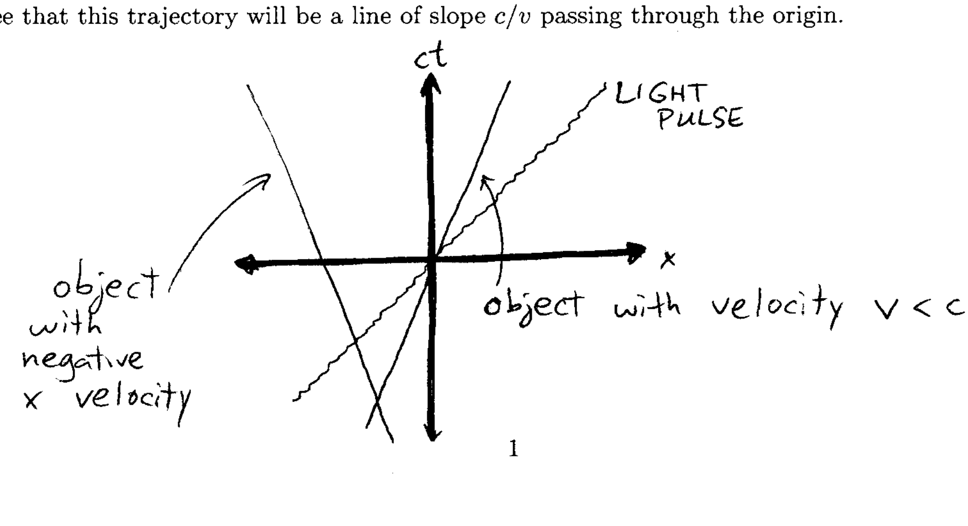
The basic things that we want to represent in spacetime diagrams are EVENTS (x, t), which happen at a particular location and a particular time, and TRAJECTORIES $x(t)$ where a physical object moves along the x axis as a function of time and is at location $x(t)$ at time t. When discussing spacetime diagrams, we usually just assume that all the motion is along one direction, so we can take $y = z = 0$ for all the events or trajectories. The spacetime diagram is then just a graph of position (horizontal axis) versus time (vertical axis). It is perhaps a little more common to thing of time as the horizontal axis, but conventionally, spacetime diagrams are drawn with time being vertical, as shown below:



Since events happen at a particular location at a particular time, these appear as points on the diagram. On the other hand, a physical object is something that has some location $x(t)$ for any time, so the trajectory of such an object will appear as a curve on the diagram, with a unique x value (the location of the object) for each time value. For example, a stationary object will be represented by a vertical line, since it has the same position for all times. An object moving at a constant velocity will have a positive slope, since it's x position increases with time. When drawing the spacetime diagram, it is conventional to choose units so that light pulses (which travel at c) will appear as 45 degree lines on the diagram. One way to do this is just to plot ct (time multiplied by the speed of light) along the vertical axis, rather than just t. Then, provided we choose the same units for x and ct, the trajectory of a light pulse $x = x_0 \pm ct$ will be a line of slope ± 1. For an object traveling at a constant velocity v that passes through $x = 0$ at $t = 0$, we have $x = vt$. To see how this looks on the diagram, we need to remember that the vertical axis is ct, so really we want to write ct in terms of x, and then make a graph in the usual way. For $x = vt$, we can rearrange to find

$$ct = \frac{c}{v} x \, ,$$

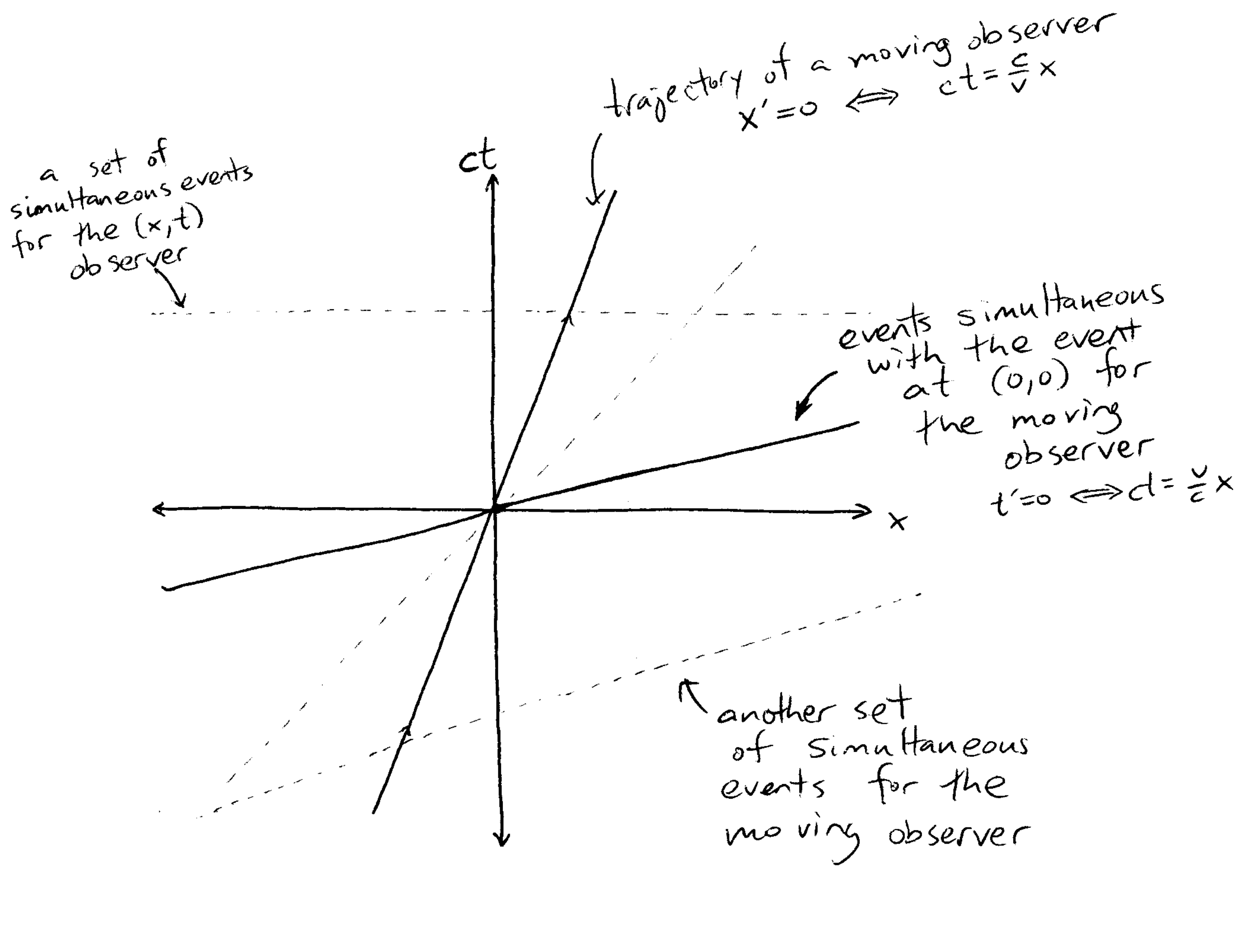
so we see that this trajectory will be a line of slope c/v passing through the origin.



Simultaneous events

On a spacetime diagram, simultaneous events are events that sit on the same horizontal line: these lines are lines of constant t, so any two events on such a line will have the same time t, and thus be simultaneous for all observers in this frame. Now, suppose there is an observer moving at a constant velocity v in the x direction. Which events will be simultaneous for this observer? To answer this, we can use the Lorentz transformation. For any event at position x and time t, the moving observer will measure that event to have position $x' = \gamma(x - vt)$ and time $t' = \gamma(t - v/c^2 x)$. Simultaneous events for the moving observer will be events that she measures to have the same t'. For example, events that the moving observer measures to be simultaneous with the event at the origin of coordinates will have $t' = 0$, but since $t' = \gamma(t - v/c^2 x)$, this will be true if and only if $ct = \frac{v}{c}x$. Thus, the set of events with $t' = 0$ appears on our diagram as a line of slope v/c, as shown below. Note that this line is just the reflection of the line representing the observer's trajectory in the 45 degree line. The set of events that occur at any other value of t' will appear on the diagram as a line parallel to the $t' = 0$ line. It must be parallel, since if the two lines intersected, the point at the intersection would have two different values of t', which makes no sense.

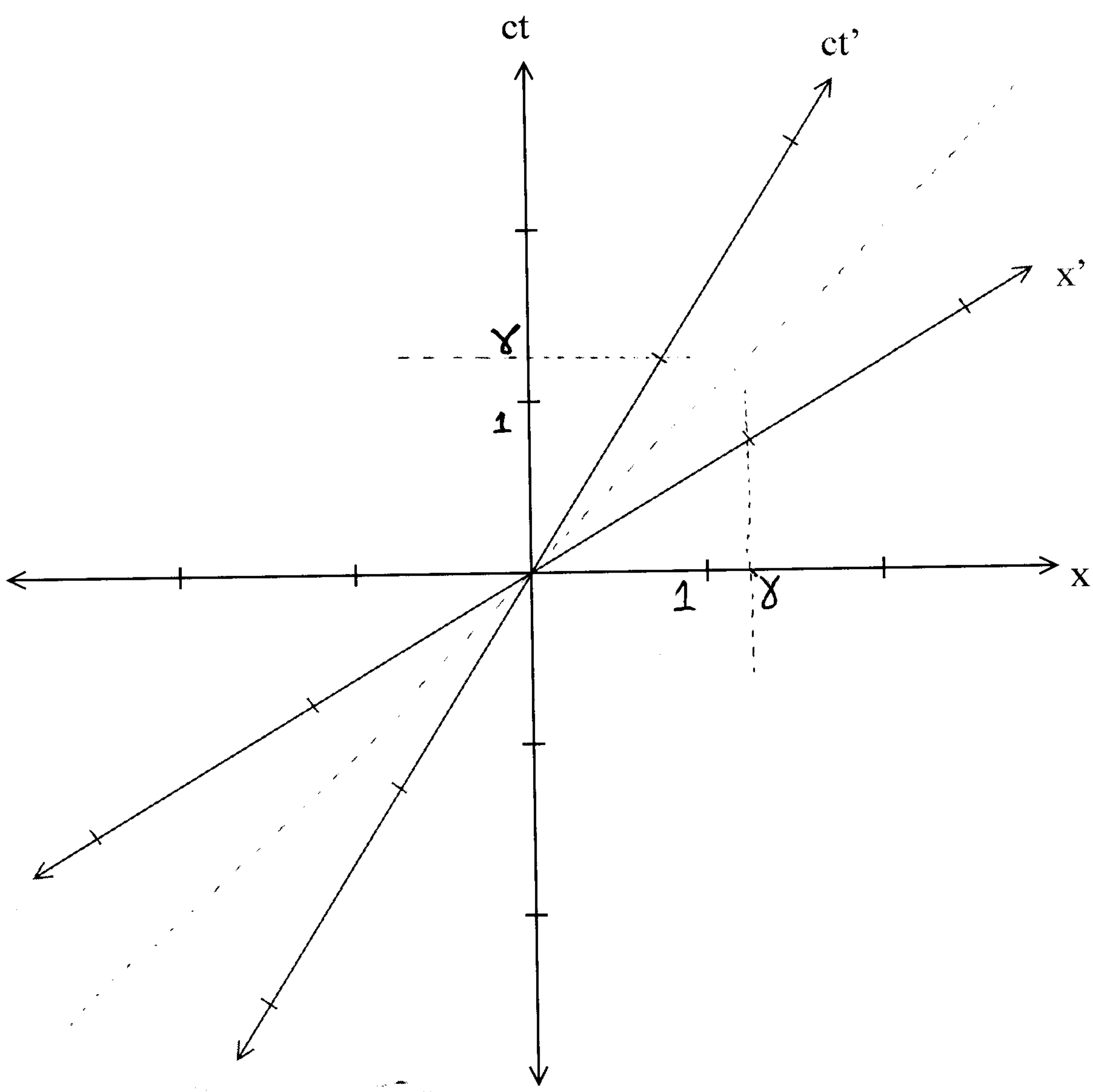
In a similar way, events which are at the same place in the frame of a moving observer will lie on a line of constant x'. All such lines are parallel to the line $x' = 0$ which is just the trajectory of the observer.



The x' and ct' axes

The x axis on our spacetime diagram is the set of events for which $t = 0$, while the t axis on the diagram is the set of points where $x = 0$. In the same way, we can call the set of points where $x' = 0$ (i.e. the trajectory of the observer with velocity v passing through the origin) the t' axis, and the set of points with $t' = 0$ (shown in the diagram above) the x' axis. So we see that the Lorentz transformation (for positive v) squashes the coordinate axes down towards the line of slope one in a symmetrical way. One more thing we need to understand is where to put the ticks on the new coordinate axes.

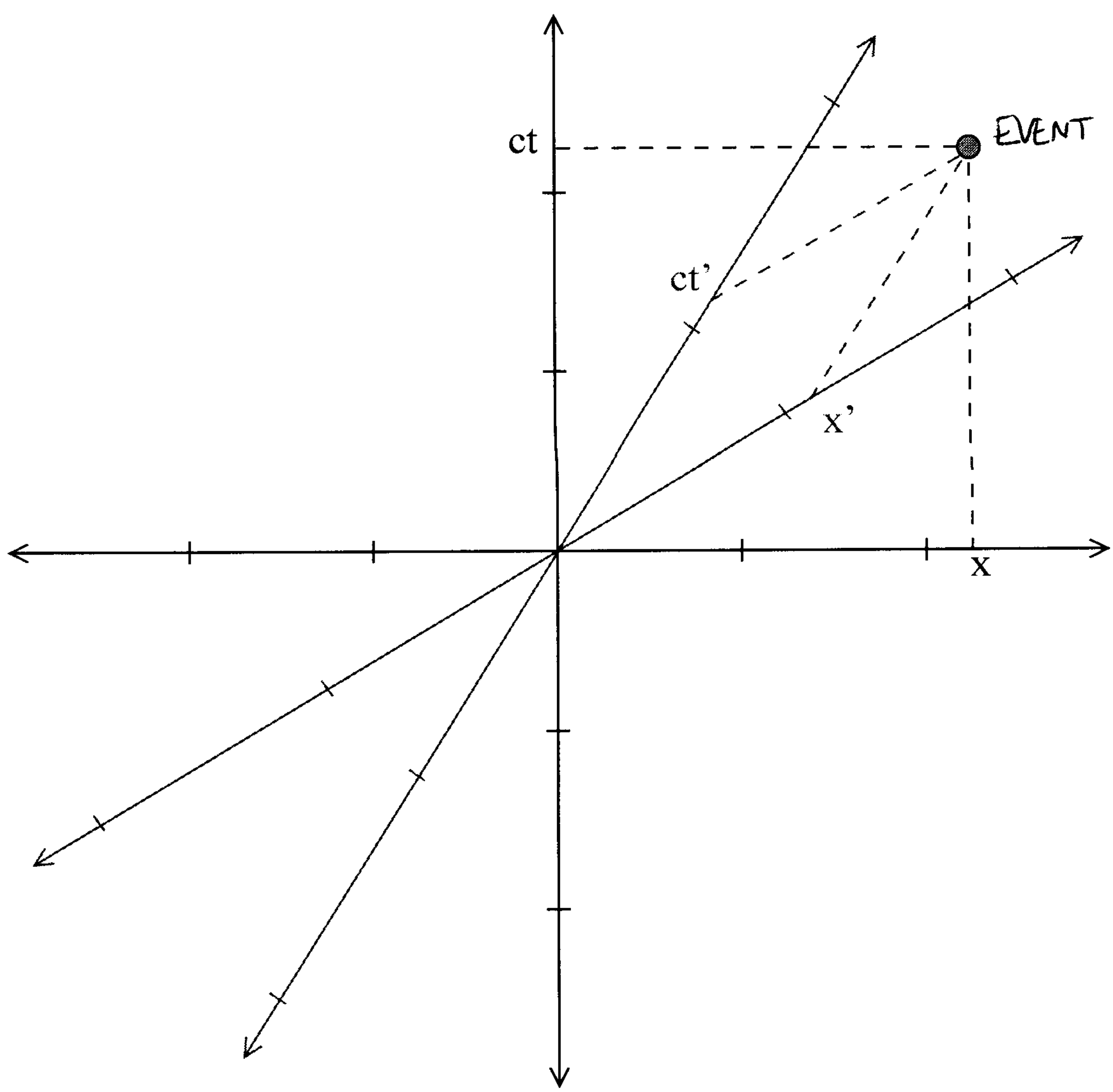
Suppose we have ticks on the x and ct axes corresponding to 1 light year intervals, and we want to put ticks on the x' and ct' axes at the same intervals. The tick at one light year on the ct' axis is the event where the moving observer's clock reads one year. Since the moving observer's clock appears to run slow by a factor of γ, this tick should be at a point with $t = \gamma$ years. Similarly, by length contraction, a stationary object of length γ in the (x, t) frame will appear to have length 1 in the (x', t') frame, so the tick at one unit along the x' axis will be at γ units along the x axis, as shown below.



The x' and ct' axes II

Now that we have ticks on both the (x, t) axes and the (x', t') axes, we can quickly find either the (x, t) coordinates or the (x', t') coordinates of any event by looking at the diagram.

When finding the x' and t' coordinates, we have to be careful, since these axes do not appear to be perpendicular on the diagram. To find the x' coordinates of an event, we draw a line of constant x' (parallel to the t' axis) through the event and see where it intersects the x' axis. To find the t' coordinate, we draw a line of constant t' (parallel to the x' axis) through the event and see where it intersects the t' axis.



Example

As a final example, let's use a spacetime diagram to see how it is possible that two observers with relative velocity will each see the other's clock run slow. From the diagram, we see that the event when the moving observers clock reads 1 (event B) is at the same time in the (x, t) frame as the "fixed" observer's clock reading γ. So the fixed observer sees the moving observer's clock run slow.

But for the moving observer, the event A where the fixed observer's clock reads 1 is at the same time (i.e. the same t' as the event where the moving observers clock reads γ. So both observers see the other's clock read 1 when their own clock reads γ. From the diagram, we see that the resolution of this apparent paradox has to do with the fact that the notion of simultaneous is different for the two observers.