Lorentz transformation:
Let's have two reference frames, A and B. Frame B moves w.r.t, A with velocity v. Assume the origins of the two frames lined up at t=0 in both frames.

An event E has coordinates x_A and t_A in frame A

What are its coordinates in frame B? the answer is the Lorentz transformation.

First of all, if $x_A = 0$ AND $t_A = 0$ THEN

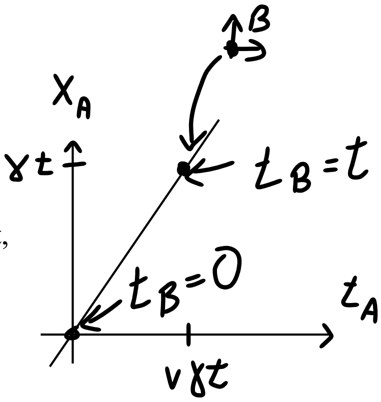
$x_B = 0$ AND $t_B = 0$ by assuption above.

Second of all, we know that on spacetime diagrams, lines (objects moving with constant velocity) go to other lines (objects moving with constant velocity).
This means that the transformation must be a linear function, like this:

$$x_B = \alpha x_A + \beta t_A + \gamma$$
$$t_B = \delta x_A + \varepsilon t_A + \kappa$$

PUT IN $x_A = t_A = 0 \Rightarrow \gamma = \kappa = 0$

$$\Rightarrow \begin{cases} x_B = \alpha x_A + \beta t_A \\ t_B = \delta x_A + \varepsilon t_A \end{cases}$$



Take a clock at the origin of frame B. The moving clock runs slow, when it shows t,

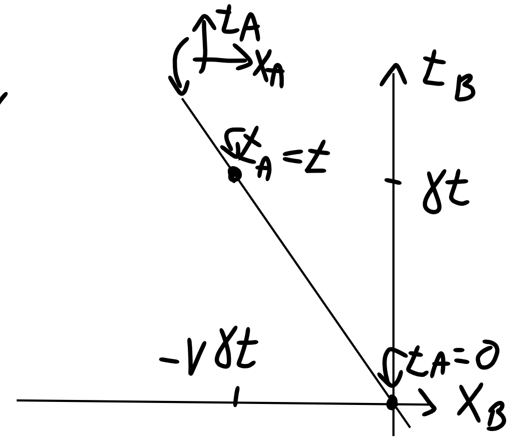
$$t_A = \gamma t \quad \text{AND} \quad x_A = \gamma t v$$

$$\Rightarrow \begin{cases} 0 = \alpha \gamma t v + \beta \gamma t & \Rightarrow \beta = -v\alpha \\ t = \delta \gamma t v + \varepsilon \gamma t & \Rightarrow \varepsilon = -v\delta + \frac{1}{\gamma} \end{cases}$$

Plugging this in, we get

$$\begin{cases} x_B = \alpha x_A - \alpha v t_A \\ t_B = \delta x_A + \left(-\delta v + \frac{1}{\gamma}\right) t_A \end{cases}$$

Now consider the clock at the origin in frame A. That clock is moving with -v, and also is time-dilated wrt clocks in B:



$$\begin{cases} -v\gamma t = \alpha \cdot 0 - \alpha v t \Rightarrow \alpha = \gamma \\ \gamma t = \delta \cdot 0 + \left(\frac{1}{\gamma} - v\delta\right) t \end{cases}$$

$$\Downarrow$$

$$\gamma = \frac{1}{\gamma} - v\delta$$

$$-v\delta = \frac{\gamma^2 - 1}{\gamma} = \frac{\frac{1}{1-v^2/c^2} - \frac{1-v^2/c^2}{1-v^2/c^2}}{\gamma} = \frac{v^2/c^2}{(1-v^2/c^2)\frac{1}{\sqrt{}}} = \frac{v^2/c^2}{\sqrt{}} = \gamma \frac{v^2}{c^2}$$

$$\delta = -\gamma \frac{v}{c^2}$$

$$\Rightarrow \begin{cases} x_B = \gamma x_A - \gamma v t_A \\ t_B = -\gamma \frac{v}{c^2} x_A + \gamma t_B \end{cases}$$

$$\boxed{\begin{aligned} x_B &= \gamma (x_A - v t_A) \\ t_B &= \gamma \left(t_A - \frac{v}{c^2} x_A\right) \end{aligned}}$$

-> Clicker question

Inverse:

$$x_A = \gamma \left(x_B + v t_B\right)$$

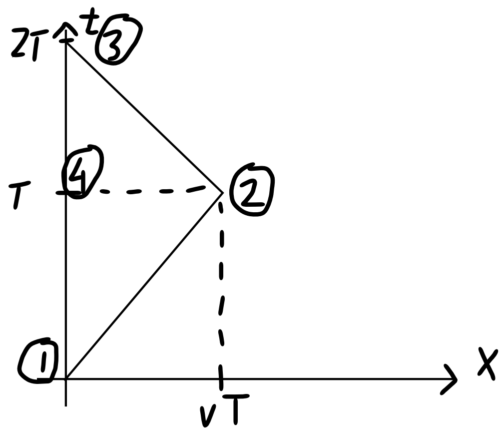
$$t_A = \gamma \left(t_B + \frac{v}{c^2} x_B\right)$$

Let's move onto length contraction... we derived this in the tutorial. A different way to derive it is using the LT.

-> clicker question

The twin paradox

In the Earth's frame, a ship goes on a round trip, with speed v:



To write things down from the ship's point of view, we need 2 frames of reference, since the ship's velocity changes midway.

Frame 1 - moves with +v wrt the Earth
Frame 2 - moves with -v wrt the Earth

Lorentz transformations:

$$\begin{cases} x_1 = \gamma(x - vt) \\ t_1 = \gamma\left(t - \frac{v}{c^2}x\right) \end{cases} \qquad \begin{cases} x_2 = \gamma(x + vt) \\ t_2 = \gamma\left(t + \frac{v}{c^2}x\right) \end{cases}$$

	(t, x)	(t_1, x_1)
1	$(0,0)$	
2	(T, vT)	*
4	$(T, 0)$	

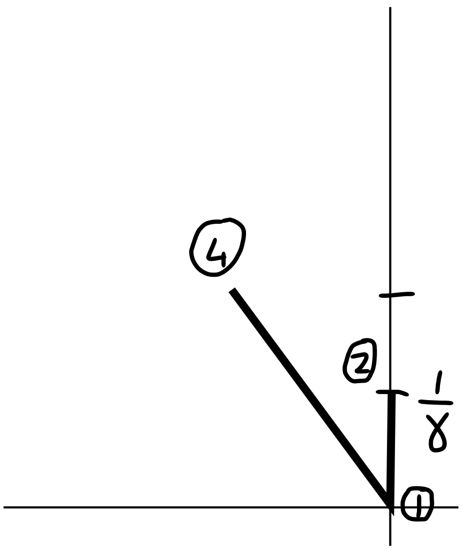
	(t, x)	(t_2, x_2)
4	$(T, 0)$	
2	(T, vT)	
3	$(2T, 0)$	

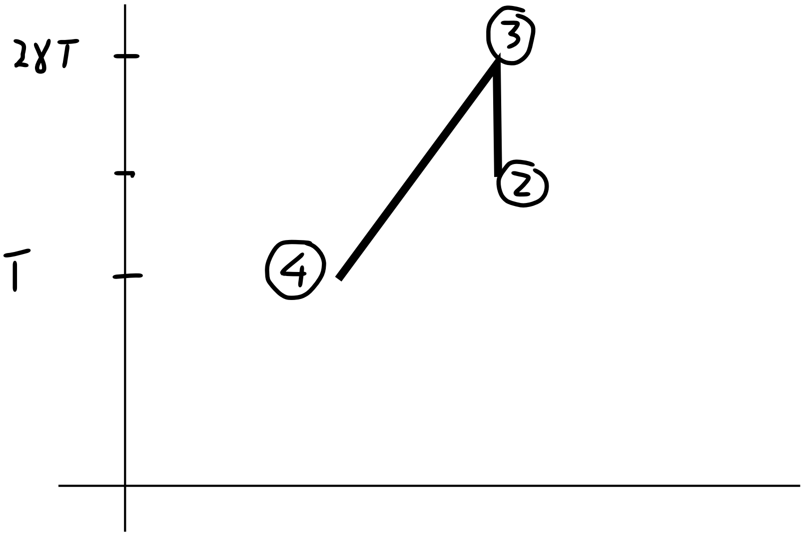
-> clicker question *

	(t, x)	(t_1, x_1)
1	$(0,0)$	$(0,0)$
2	(T, vT)	$(\frac{1}{\gamma}T, 0)$
4	$(T, 0)$	$(-\gamma T, -\gamma vT)$

	(t, x)	(t_2, x_2)
4	$(T, 0)$	$(\gamma T, -\gamma vT)$
2	(T, vT)	$(\gamma(1+\frac{v^2}{c^2})T, 2\gamma vT)$
3	$(2T, 0)$	$(2\gamma T, 2\gamma vT)$

Claim: $\gamma\left(1+\frac{v^2}{c^2}\right) = 2\gamma - \frac{1}{\gamma}$





Vertical is the ship, at an angle is the Earth

Notice the 'gap': when ship changes reference frames, the clocks it's comparing itself againts 'jump'. This is why the ship does not age as much as the Earth does.

Compare this to the naive (WRONG) picture, where the Earth goes back and forth and the ship stays.