In the tutorial yesterday (Tutorial 10), you saw that wavepackets evolve: they move, they spread, they change shape.

A simple de Broglie wave has simple time dependence. It is given by the de Broglie frequency:

$$e^{i\frac{2\pi}{\lambda}x - i2\pi f t} = e^{\frac{2\pi i}{h}(px - Et)}$$

$$\lambda = h/p \qquad f = E/h$$

How fast does such a wave move? we know that $v_{\text{wave propagation}} = \lambda f = E/p = \frac{\frac{1}{2}mv^2}{mv} = \frac{1}{2}v$

The wavefronts move with half the expected speed. This is known as PHASE VELOCITY.

As you saw in the tutorial, the above formula for f ensures that **wavepackets** with average momentum p move with speed v=p/m. The wavepacket moves with the same speed the particle does. The speed of the wavepacket is called the GROUP VELOCITY.

-> look at the simulation of wave packets to see that the phase velocity is smaller than the group velocity.

Since we know that all wavefunctions can be expanded as a sum over planewaves, this is enough to figure out how any wavefunction changes with time:

Say, at t=0, we have

$$\psi(x, t=0) = \int \tilde{\psi}(p)\, e^{i\frac{2\pi}{h}xp}\, dp$$

QM is a **linear theory**: the principle of superposition holds. This means that each component of the wave evolves in time independently of all the other componets. Therefore:

$$\psi(x, t) = \int \tilde{\psi}(p)\, e^{\frac{2\pi i}{h}(xp - Et)}\, dp$$

How does the momentum wavefunction change?
-> clicker question

$$\psi(x, t) = \int \tilde{\psi}(p, t)\, e^{\frac{2\pi i}{h}xp}\, dp$$

$$\Rightarrow \quad \tilde{\psi}(p, t) = \tilde{\psi}(p)\, e^{-\frac{2\pi i}{h}Et}$$

A useful way to describe the time evolution of a function is often through a differential equation. Start with the following observation:

$$E = \frac{1}{2}mv^2 = \frac{1}{2}\frac{p^2}{m}$$

Notice that you can 'pull down' a factor of E from a planewave by acting on it with a time derivative

$$\frac{\partial}{\partial t} e^{-i\frac{2\pi}{h}Et} = -i\frac{2\pi}{h} E e^{-i\frac{2\pi}{h}Et}$$

$$i\hbar \frac{\partial}{\partial t} e^{-i\frac{Et}{\hbar}} = E\, e^{-iEt/\hbar}$$

LET $\hbar = \frac{h}{2\pi}$

pronounced h-bar

You can also pull down a factor of p with an x-derivative:

$$-i\hbar \frac{\partial}{\partial x} e^{ixp/\hbar} = p\, e^{ixp/\hbar}$$

So, we can write

$$E\, e^{\frac{i}{\hbar}(xp-Et)} = \frac{1}{2m} p^2 e^{\frac{i}{\hbar}(xp-Et)}$$

$$i\hbar \frac{\partial}{\partial t} e^{\frac{i}{\hbar}(xp-Et)} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} e^{\frac{i}{\hbar}(xp-Et)}$$

By linearity, then:

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t)$$

This is the (time-dependent) Schrodinger equation for a free particle.
What does this equation tell us? we can write it as

$$\frac{\partial}{\partial t}\psi = \frac{i\hbar}{2m} \partial_x^2 \psi$$

$$\psi(x,t+\Delta t) \simeq \psi(x,t) + \Delta t\, \partial_t \psi(x,t) = \psi(x,t) + \overbrace{\frac{i\hbar \Delta t}{2m} \partial_x^2 \psi}^{\Delta \psi}$$

You can determine the wavefunction at position x and time t+Δt from the properties of the wavefunction at time t and position x

-> clicker question

We already know what happens to wavepackets under this equation: they move with a group velocity equal to the average momentum divided by mass, they spread and even change shape.

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