Last time: position wavefunction $\psi(x)$ and momentum wavefunction $\tilde{\psi}(p)$

Probability density of detecting a particle at position x = $|\psi(x)|^2$

Probability density of detecting a particle having momentum p = $|\tilde{\psi}(p)|^2$

Related by a fourier transform:

$$\psi(x) = \int \underbrace{\tilde{\psi}(p)}_{\substack{\text{coefficient determines}\\ \text{how much of the mode}\\ \text{with momentum p}\\ \text{contributes to the total}\\ \text{wavefunction}}} \underbrace{e^{i\frac{2\pi}{h}px}}_{\substack{\text{plane wave with}\\ \text{momentum p}}} \, dp$$

We already know that wider momentum distribution is needed for a narrow wavepacket, etc... but want to make this qualitative.

We would like to have some way of talking about the average position or momentum, and the uncertainity in position and momentum.

Let's consider an example with a die throw.

-> clicker question

Notice that we can compute the average result by multiplying each possible outcome (1, 2, 3 etc...) by its probability (1/6):

$$\text{AVERAGE} = \frac{1}{\#\text{ of throws}}\left((1)\begin{pmatrix}\text{number of times}\\ \text{we get a 1}\end{pmatrix} + (2)\begin{pmatrix}\text{number of times}\\ \text{we get a 1}\end{pmatrix} + \ldots\right)$$

$$= (1)\begin{pmatrix}\text{probability of}\\ \text{getting a 1}\end{pmatrix} + (2)\begin{pmatrix}\text{probability of}\\ \text{getting a 2}\end{pmatrix} + \ldots$$

$$= 1(1/6) + 2(1/6) + \ldots = 21/6 = 3{,}5$$

By analogue, to compute the average position of a particle, we can compute

$$\langle x \rangle = \underbrace{\int}_{\text{sum}} \underbrace{|\psi(x)|^2}_{\text{probability}} \underbrace{x}_{\text{outcome}} \, dx$$

Or, for the average momentum,

$$\langle p \rangle = \int |\tilde{\psi}(p)|^2 \, p \, dx$$

The interpretation of these averages is that if you were to make, say, a position measurement on **many particles prepared to be in the same wavefunction**, the average of all of these measurements would be $\langle x \rangle$

What about uncertainity?

Let's go back to the die example. We could measure how uncertain we are about any particular throw being close to the average by looking at many throws and writting down how far from the average they are

throw result	devation from average
3	-0.5
6	2.5
4	0.5
2	-1.5

To make sure that the positive deviations don't cancel out the negative deviations, let's square them before adding up, weighted by the probabilities, and then take a square root. This is called the **mean deviation:**

$$\sqrt{\left(\begin{smallmatrix}\text{probability}\\ \text{of 1}\end{smallmatrix}\right)(1-3.5)^2 + \left(\begin{smallmatrix}\text{probability}\\ \text{of 2}\end{smallmatrix}\right)(2-3.5)^2 + \dots}$$

$$= 1.7$$

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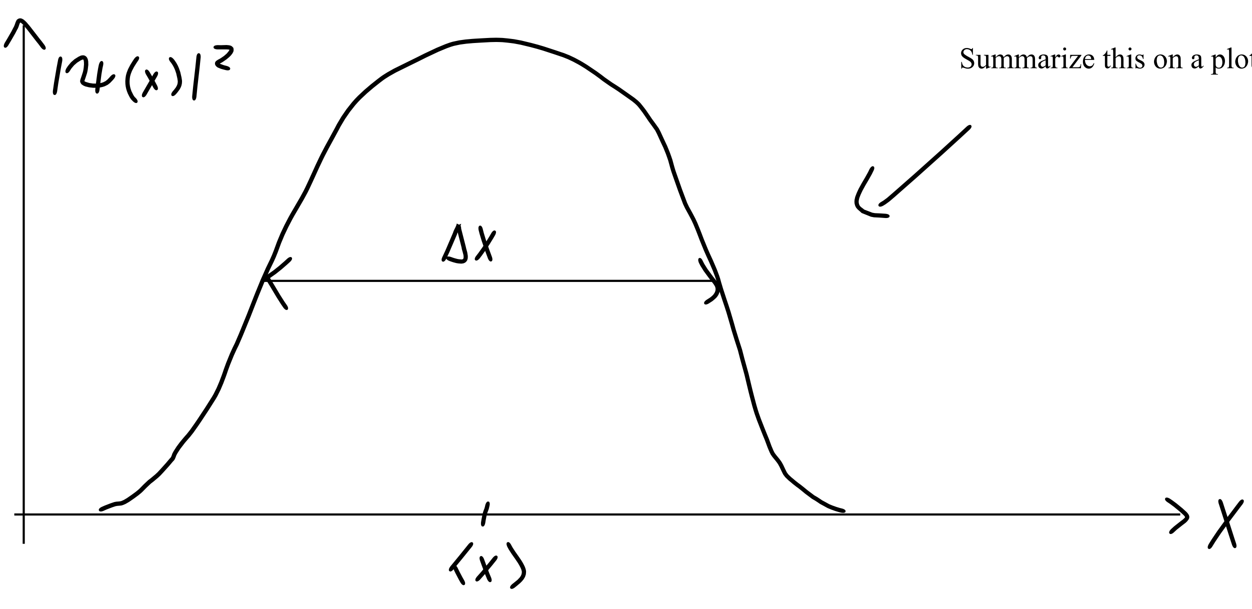
The uncertainty with a 1,1,1,6,6,6 die is bigger: all deviations are either -2.5 or 2.5, so the uncertainty is 2.5
The mean deviation is a measure of how close to the average we expect most measurements to be.

We can also compute the uncertainty of a position measurement:

$$\Delta x \stackrel{\text{def}}{=} \sqrt{\int |\psi(x)|^2 \left(x - \langle x \rangle\right)^2 dx}$$

$$\Delta p \stackrel{\text{def}}{=} \sqrt{\int |\psi(p)|^2 \left(p - \langle p \rangle\right)^2 dp}$$

Summarize this on a plot:



There is a mathematical theorem (in Fourier theory) that

$$(\Delta x)(\Delta p_x) \geq \frac{h}{4\pi}$$

h is very, very small, so for macroscopic objects, this uncertainty is never important: if you measure the position and momentum of a baseball as accuratelly as humanly possible, $(\Delta x)(\Delta p)$ will be much bigger than h.

In physics this is known as the **Heisenberg uncertainty principle**.

No state can have a definite position and momentum.
For example, if you measure the electron's position with (experimental) uncertainty (error) Δx , it's momentum has to become uncertain by an amount at least as large as $\frac{h}{4\pi} \frac{1}{\Delta x}$

Notice that this is a lower limit. Uncertainties can be bigger than this, but not smaller.
- > clicker question

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That last clicker question is bringing us up to the problem how time evolution works for wavefunctions.

When De Broglie postulated his matter waves, he postulated not only a wavelength $\lambda = h/p$

but also a frequency $f = h/E$ where E is simply the energy of the particle.

With these two pieces of information, we can write down how a plane wave behaves for all times:

$$e^{i\left(\frac{2\pi}{\lambda} x - 2\pi f t\right)} = e^{i \frac{2\pi}{h}(px - Et)}$$

What about more general wavepackets?
Decompose them into the planewave eigenstates of momentum and let each eigenstate evolve according to de Broglie's formula

$$\psi(x) = \int \tilde{\psi}(p) \, e^{i \frac{2\pi}{h}(px)}$$

$$\downarrow$$

$$\psi(x,t) = \int \tilde{\psi}(p) \, e^{i \frac{2\pi}{h}(px - Et)}$$

This leads to all sorts of peculiar behaviours. In tomorrow's tutorial, you will play with a symulation of the time evolution of wavepackets, and then derive de Brogie's formula for the frequency given above.

On Friday, we will use this approach to derive the Schrodinger equation,