Notes on complex numbers

A complex number is defined as $a+bi$ where a and b are real numbers. The symbol i has the (defining) property that $i \times i = -1$. Ordinary rules of algebra apply to complex numbers: you can treat i as any other 'variable' as long as you use $i^2 = -1$ to simplify expressions.

a is called the real part of the complex number $a+ib$. This is often writen as $a = \Re(a+ib)$ or $a = \mathrm{Re}(a+ib)$. b is called the imaginary part of $a+ib$, with notation $b = \Im(a+ib)$ or $a = \mathrm{Im}(a+ib)$.

An extremely useful fact (proven briefly in Tutorial 8) is the following:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

[Aside: this is true even if θ itself is a complex number.]

Often it is useful to represent complex numbers in the so-called polar form, $re^{i\theta}$, where r is real and non-negative and θ is real. r is called the magnitude of the complex number and θ its phase. Using that $e^{i\theta} = \cos\theta + i\sin\theta$, we get the following relationships:

$$\text{If } a + ib = re^{i\theta}, \text{ then :}$$

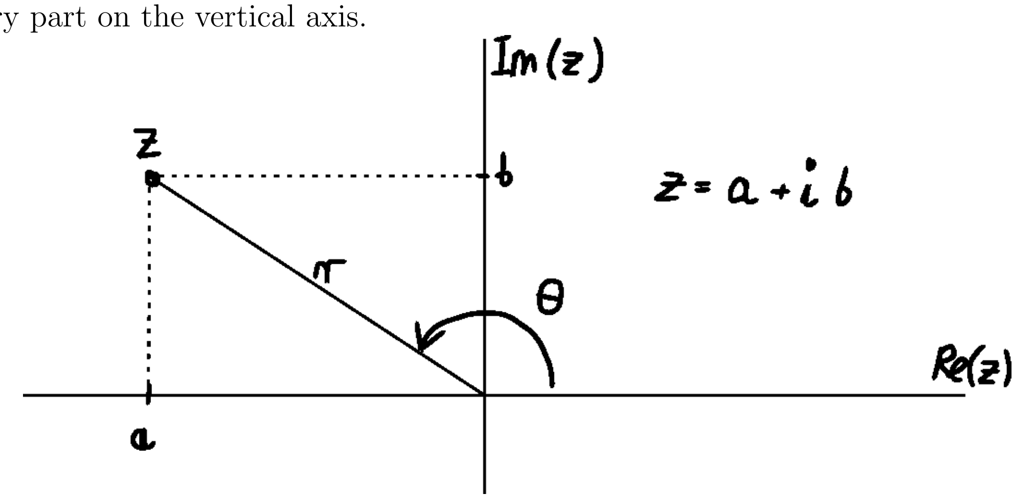
$$a = r\cos\theta$$

$$b = r\sin\theta$$

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}(b/a) \text{ (make sure you get the right quadrant!)}$$

The above relationships are usefully pictured by plotting a complex number in the complex plane: a two-dimensional plane with the real part of the number on the horizontal axis and the imaginary part on the vertical axis.



Practice problems

1. Simplify (write in the form $a + ib$) the following complex numbers (no calculators):

$i^{1001} =$

$e^{i\pi/4+\ln(3)} + (3+2i)^3 - (e^{i\pi}+1)^{100} =$

$(\sin(\pi/100) + i\cos(\pi/100))^{100} =$

$\sqrt{i} =$ (hint: write i in polar form)

2. Compute the magnitude and the phase of (still no calculators)

$(\sqrt{3}+i)^{10}$ (hint: convert to polar form before taking the power)

$(5e^{i\pi/4} - 5i/\sqrt{2})^3$

3. Challenge question: what is $\cos(\pi/3 + i\pi/4)$?