Physics 200 Problem Set 10

Due at the end of class, Fri Nov 26th

Practice problems: do not hand in. Textbook pg 1259-1260, problems 23, 25, 41, 43.

1. Consider an electron which at time $t = 0$ is located at $x = 0$ and whose mean momentum is 0 as well. The normalized wavefunction at t=0 is given by

$$\Psi(t=0,x) = \frac{1}{\pi^{1/4}\sqrt{L_0}} \exp\left(-\frac{x^2}{2L_0^2}\right)$$

(a) Plot (or sketch) the wavefunction for L_0=1, 4 and 16. What does L_0 correspond to?

(b) We want to solve the time-dependent Schrodinger equation for a free particle, with the above wavefunction as the initial condition. Let's guess that the wavepacket will keep its overall shape but change its width and normalization with time. Take the ansatz wave-function

$$\Psi(t,x) = \frac{1}{\pi^{1/4}\sqrt{L_0 f(t)}} \exp\left(-\frac{x^2}{2L_0^2 g(t)}\right)$$

and substitute it into the time-dependent Schrodinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$$

to obtain the following equation:

$$\frac{g'(t)}{g(t)^2}\frac{x^2}{L_0^2} - \frac{f'(t)}{f(t)} = \frac{i\hbar}{m}\left(\frac{1}{g(t)^2}\frac{x^2}{L_0^4} - \frac{1}{g(t)L_0^2}\right)$$

(c) Argue that f(0)=1 and g(0)=1.

(d) In the equation you obtained in part (b), take $x \to \infty$ and obtain $g(t)$ from the resulting equation.

(e) What is f(t)?

2. In the first question, you derived that

$$\Psi(t,x) = \frac{1}{\pi^{1/4}}\sqrt{\frac{L_0}{\left(L_0^2 + \left(\frac{i\hbar}{m}\right)t\right)}}\ \exp\left(-\frac{x^2}{2\left(L_0^2 + \left(\frac{i\hbar}{m}\right)t\right)}\right)$$

is a solution to the time-dependent Schrodinger equation. In this question we will try to understand the shape of this wavefunction.

(a) In the wavepacket simulation we used for Tutorial 10:
http://phet.colorado.edu/en/simulation/quantum-tunneling
set up a wavepacket with zero energy in a constant potential and an initial width of 0.3nm. Sketch the shape of the real part of the wavefunction and the magnitude, as shown by the simulation, at t=0, 3 and 6 fs.

(b) To see why the wavefunction develops oscillatory behaviour as it evolves in time, we can ignore the normalization factor in $\Psi(x,t)$, as it will not affect the shape since it does not depend on x. Let's focus on the argument of the exponent in the wavefunction above:

$$-\frac{x^2}{2\left(L_0^2 + \frac{i\hbar}{m}t\right)}$$

Multiply the top and the bottom of this expression by $L_0^2 - i\hbar t/m$ and write the result in the form $a + ib$ with a and b real.

(c) Use your answer to part (a) together with $\exp(a+ib) = \exp(a)(\cos(b) + i\sin(b))$ to compute the real part of

$$\exp\left(-\frac{x^2}{2\left(L_0^2 + \frac{i\hbar}{m}t\right)}\right)$$

(d) Which part of your expression in part (c) is responsible for the wavepacket getting wider, and which is responsible for the oscillations?

Recall that in class, we discussed the following formulas for mean or average x, $\langle x, \rangle$ and the mean deviation in x, Δx:

$$\langle x \rangle = \int |\psi(x)|^2 \ x \ dx$$

$$(\Delta x)^2 = \int |\psi(x)|^2 \ (x - \langle x \rangle)^2 \ dx$$

3. An electron has a wavefunction given by

$$\psi(x) = \frac{A}{a^2 + x^2}$$

(a) What is A if the wavefunction is normalized? [Hint: You might find $x = a \tan y$ to be a useful substitution.]

(b) If position was measured repeatedly for many electrons with the above wavefunction, what would be the average outcome?

(c) What is the the mean deviation for the measurements in part (b)?

4. In this question, we will again consider the wavefunction from Question 3, but this time we will talk about it in momentum space.

(a) We know that a wavefunction in position space, $\psi(x)$, can be written as a combination of momentum eigenstates, i.e. plane waves $e^{ixp/\hbar}$, with various momenta p. Formally, we have

$$\psi(x) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} \tilde{\psi}(p) \ e^{ixp/\hbar} \ dp$$

To write the inverse relationship, we need to use the Fourier Transform formula, which tells us that the above equation is equivalent to

$$\tilde{\psi}(p) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} \psi(x) \ e^{-ixp/\hbar} \ dx$$

Using the following 'math fact':

$$\int \frac{e^{ikx}}{1 + x^2} \ dx = \pi e^{-|k|}$$

compute the momentum wavefunction $\tilde{\psi}$ corresponding to the wavefunction given in Question 3.

(b) What is the mean momentum for this particle?

(c) What is the mean deviation in momentum, Δp?

(d) Does the product $\Delta x \Delta p$ satisfy the Heisenberg inequality?

(e) Let $a = 1\text{nm}$. If we measure the velocity of this electron, what is the probability that we will find the speed be greater than 10^6m/s?