Vortex Structures in Neutron Stars

or How I Learned to Stop Worrying and Just Write It Already

by

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Abstract

Every day I get sucked into a freaking vortex and crash-land my freaking space-ship on my own freaking helmet. Because of this my bald spot is getting worse, as evidenced by the increasing number of people who like to point it out. This process has been dramatized in the medium of sequantial art (fig.1) by Nicholas Gurewitch of the Perry Bible Fellowship [4].

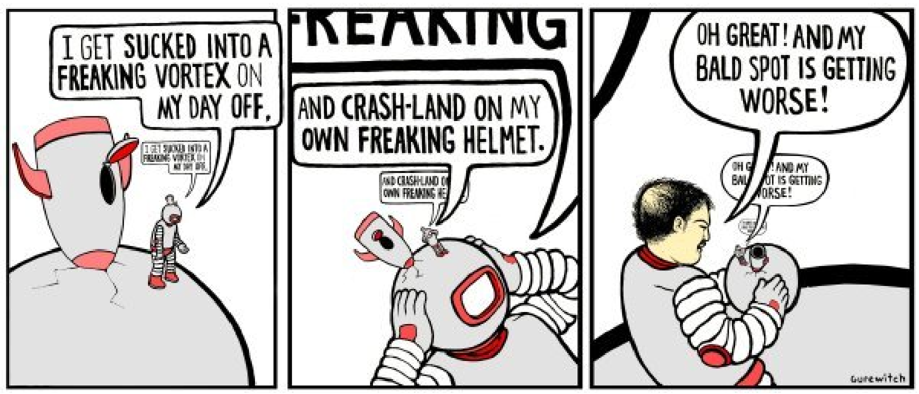

Figure 1: Freaking Vortex

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In My Craft or Sullen Art

In my craft or sullen art
Exercised in the still night
When only the moon rages
And the lovers lie abed
With all their griefs in their arms,
I labour by singing night
Not for ambition or bread
Or the stut and trade of charms
On the ivory stages
But for the common wages
Of their most secret heart.

Not for the proud man apart
From the raging moon I write
On these spindrift pages
Nor for the towering dead
With their nightingales and psalms
But for the lovers, their arms
Round the griefs of the ages,
Who pay no praise or wages
Nor heed my craft or sullen art.

Dylan Thomas

Part I

Review of Standard Theory

Chapter 1

Introduction

The motivation for this thesis comes from the observation that neutron stars precess [13] and from a calculation by Link [22] which shows that the magnitude and frequency of the precession contradicts the standard model of a neutron star. It is the purpose of this introduction to clarify this motivation.

Neutron stars have been a topic of great interest since it was identified that pulsars were actually rotating neutron stars that emit electromagnetic radiation in pulses. A neutron star is a very extreme object. They are incredibly dense, and generate huge magnetic fields, and yet are incredibly cold. Measuring the time between the electromagnetic pulses gives us an idea of how quickly the star rotates, but it was quickly recognized that they do not spin how we expect. Instead, the timing of these pulses indicates that the star will suddenly speed up, an event known as a glitch, then slowly return to it's original rate. It is these conditions which make neutrons stars so interesting to study.

It is believed that the answers to these questions can be found in laboratories here on earth. Explorations of condensed matter at cold temperatures have lead to a number of phenomena, the most interesting being Bose/Fermi condensation, the tendency of cold particles to all occupy the same energy state. Charged particles form condensates called superconductors and neutral particles form superfluids.

A superconductor placed in a magnetic field will act as a perfect diamagnet, repelling the field up to a critical value where defects form in the superconductor parallel to the field, allowing the magnetic flux to penetrate. A similar phenomenon occurs when a superfluid is subjected to angular momentum. Initially, because the superfluid has no viscosity, the superfluid will not rotate, but at a critical angular velocity a defect forms parallel to the angular velocity which carries a quanta of circulation, and the fluid rotates.

Both of these defects are called vortices. In a superfluid the vortices repel each other and form an array. Superconducting vortices can either attract each other, forming large vortices, or repel each other and form an array. These are called type I and type II superconductors respectively. A detailed review of vortices is given in chapters 2 and 3.

The cold, dense nature of a neutron star suggests that a neutron star is made of superfluid neutrons and superconducting protons, both of which form vortices. The proton vortices are higher density than the neutron vortices and entangle around them meaning that wherever one set of vortices moves, the other must as well. Chapter 4 provides a detailed discussion of how condensates and vortices arise in neutron stars.

The contradiction, which is the motivation for this thesis, comes from the fact that if the star is to precess the neutron vortices must move but the proton vortices won't allow them to. Link asserts that either the superconductor inside a neutron star is of type I and the proton vortices bunch allowing the neutron vortices to move or that the superfluid neutrons and superconducting protons do no coexist. Link's argument is detailed in the beginning of part II.

Part II of this paper investigates Link's conclusions by asking how can a superconductor be type II but act like type I and how would the vortex structure inside a neutron star change? The mechanism we will use in this investigation is the recent discovery that vortices can carry electrical current along their axes. Consider two type II superconducting vortices running parallel to each other. These would normally repel. If they carry a current, similar to two wires running parallel, then there is an attractive force between them. If the current is strong enough the vortices could attract, exhibiting type I behavior.

This attraction could facilitate a number of new vortex structures. If the vortices were to form a loop, called a vorton, in an external magnetic field it would be possible for the vortex to be stable. If a number of loops exist they will tend to stack on top of one another forming a vortex sheet which carries current quite similar to a solenoid. A brief introduction to vertex sheets will be presented in chapter 6 though a detailed analysis will be left for later papers.

1.1 Spontaneous Symmetry Breaking

Spontaneous symmetry breaking is the mechanism which is responsible for the vortices reviewed in this paper. It is poor terminology in that it implies that if a Lagrangian has a symmetry, when spontaneous symmetry breaking occurs, this symmetry of the Lagrangian is somehow destroyed. The truth is that a Lagrangian having a symmetry doesn't imply that the system when viewed from its ground state has that symmetry. Consider the parity symmetry, $\psi(x) \longrightarrow -\psi(x)$, in the 1-D potential

$$U = -\frac{m^2}{2}\psi(x)^2 + \frac{\psi(x)^4}{4}\,. \tag{1.1}$$

This has minima at $\psi = \pm m$. If we look at the potential energy from the point of view of one of these minima by shifting the field, $\psi(x) \longrightarrow \sigma(x) - m$, then

$$U = m^2\sigma(x)^2 - m\sigma(x)^3 + \frac{\sigma(x)^4}{4}\,. \tag{1.2}$$

The parity symmetry seems to have disappeared, but really the symmetry is just hidden, or as Coleman [5] likes to put it, a secret. The symmetry breaking is spontaneous in the sense that the symmetic state is an unstable one, and that the system will quickly decay into a non-symmetric ground state.

The parity symmetry is a discrete symmetry. Breaking a continuous symmetry is much more interesting. Goldstone's theorem states that whenever a continuous symmetry is broken a massless, non-dissipative, mode appears. This is usually illustrated using the linear sigma model [30] and the breaking of the $SO(N)$ symmetry. Because we are working with a complex field, which can be written as $\psi = \rho e^{i\varphi}$, it would be helpful to demonstrate breaking of a $U(1)$ symmetry. The following discussion is similar to that found in [16].

Consider the energy

$$E = \int d^3x \left[\frac{1}{2}|\nabla\psi|^2 - \frac{m}{2}|\psi|^2 + \frac{g}{4}|\psi|^4 \right] . \tag{1.3}$$

This energy is has a symmetry under one dimensional unitary transformations: $\psi(x) \longrightarrow e^{iA}\psi(x)$. It is this symmetry that will be broken. By writing ψ in terms of it's magnitude and phase we break it into two fields,

$$E = \int d^3x \left[\frac{1}{2}(\nabla\rho)^2 + \frac{1}{2}\rho^2(\nabla\varphi)^2 - \frac{m^2}{2}\rho^2 + \frac{\rho^4}{4} \right] . \tag{1.4}$$

The energy is minimized when $\rho = m$. We can now look at the fluctuations of these fields around the ground state $\rho = m + \rho'$ and $\varphi = \varphi_0 + \varphi'$.

$$E = \int d^3x \left[\frac{1}{2}(\nabla\rho')^2 + \frac{m^2}{2}\rho^2 + \frac{1}{2}m^2(\nabla\varphi')^2 + (m\rho' + \frac{\rho'}{2})(\nabla\varphi')^2 + (m\rho'^3 + \rho'^4) \right] . \tag{1.5}$$

If the variations are sufficiently small then the last two terms become zero and we get the energy of the fluctuations

$$E = \int d^3x \left[\frac{1}{2}(\nabla\rho')^2 + \frac{m^2}{2}\rho^2 + \frac{1}{2}m^2(\nabla\varphi')^2 \right] . \tag{1.6}$$

We now have a Hamiltonian for our field fluctuations. Writing down the equations of motion for each field,

$$(\nabla^2 + m^2)\rho' = 0 \text{ and } \nabla^2\varphi' = 0, \tag{1.7}$$

we see that the ρ' field has a mass as expected, but the φ' field is massless. These massless modes are called Goldstone Bosons and because of the symmetry of the Lagrangian they can move freely in the $\hat{\boldsymbol{\phi}}$ direction. It is this symmetry breaking and the creation of these massless modes moving unhindered that lead to superfluid and superconducting vortices. The existence of these modes in the ground state of the system is a characteristic of non-disipative flow. Usually, such as in the case of a rotating viscous fluid, the ground state of the system is when all flow has stopped. The rotational flow will disipate to become a stationary system. This doesn't happen when the lowest energy state has flow in it, so the flow is called non-dissipative.

Chapter 2

Superconducting Vortices

In 1950 Landau and Ginzburg proposed a theory which phenomenologically describes much of the behavior seen in superconductors. Not only does it encapsulate the work done by F. London and H. London in explaining the Meissner effect, but was used to postulate some very remarkable phenomena. The focus of this chapter is Abrikosov's prediction of the vortex [1], a line defect in the superconductor which carries quantized magnetic flux. It is important to note that Ginzburg and Landau derived this theory phenomalogically, before the BCS theory of superconductivity was introduced, and that many years later Gorkov showed that it comes from BCS theory naturally.

We will start with the Landau-Ginzburg free energy and a derivation of the equations of motion [16]. In section 2.2 the equations of motion will then be used to show that the theory contains the Meissner effect [2]. Section 2.3 will discuss cylindrically symmetric solutions which lead to vortices and the quantization of magnetic flux [16] [21]. Also, the equations of motion will be solved to investigate the structure of the condensate in a vortex [15]. Using this, the energy of a single vortex [16] will be discussed in section 2.4 and the interaction energy between two vortices will be found [37] in section 2.5. This will give us insight into the stability of vortices in type I and type II superconductors. Finally, in section 2.6, the critical magnetic fields for type I [6] and type II [16] superconductors will be found.

2.1 The Ginzburg-Landau Energy

The Ginzburg-Landau energy is based on the work of Gorter and Casimir who introduced the idea of an order parameter $|\psi|^2$ proportional to the density of superconducting electrons to describe the state of a superconductor. They postulated a free energy for a superconductor near critical temperature T_c

$$E = -\mu\,|\psi|^2 + \frac{a}{2}\,|\psi|^4\ , \tag{2.1}$$

where $\mu \sim \mu_0\left(1 - \frac{T}{T_\mathrm{c}}\right)$ is the chemical potential, which changes sign at T_c, and a is related to the scattering length l, $a = \frac{4\pi\hbar^2 l}{m}$. Landau noticed this idea could be expanded by considering a complex order field $\psi(\mathbf{x})$ which could be used to describe fluctuations in the order parameter by adding a gradient to Gorter and Casimir's guess of the free energy. He and Ginzburg could then write the free energy of a superconductor near the critical temperature T_c. To investigate at a superconductors in magnetic fields, similar to F. London and H. London, they added the field energy and a gauge invariant derivative to arrive at

$$\begin{aligned} E &= \int d^2x \left\{ \frac{\hbar^2}{2m} \left| \left(\nabla - \frac{iq}{\hbar c} \mathbf{A}(\mathbf{x}) \right) \psi(\mathbf{x}) \right|^2 \right. \\ &\left. -\mu \left| \psi(\mathbf{x}) \right|^2 + \frac{a}{2} \left| \psi(\mathbf{x}) \right|^4 + \frac{1}{8\pi} \left(\nabla \times \mathbf{A}(\mathbf{x}) \right)^2 \right\} . \end{aligned} \tag{2.2}$$

We are interested in cylindrical solutions so we choose to work in 2-D, so the equation above is the energy per unit length.

This energy can be minimized to yield the Landau-Ginzburg equations. Minimizing with respect to the vector potential $\mathbf{A}$ gives us

$$\frac{1}{4\pi} \left[\nabla^2 \mathbf{A} - \nabla \left(\nabla \cdot \mathbf{A} \right) \right] = -\frac{\hbar q}{mc} \, \mathbf{j} \, , \tag{2.3}$$

where $\mathbf{j}$ is the Noether current[30]

$$\mathbf{j} = \frac{1}{2i} \left(\psi^\dagger \left(\nabla - \frac{iq}{\hbar c} \mathbf{A} \right) \psi - \left(\nabla - \frac{iq}{\hbar c} \mathbf{A} \right) \psi^\dagger \psi \right) \tag{2.4}$$

$$= \frac{1}{2i} \left(\psi^\dagger \nabla \psi - \psi \nabla \psi^\dagger \right) - \frac{q}{\hbar c} \mathbf{A} \left| \psi \right|^2 . \tag{2.5}$$

This identification of the currect is critical and will later lead to the result that flux is quantized inside a vortex. Using curl identities and the alternate form of the Noether current we can rewrite the equation of motion as

$$\frac{1}{4\pi} \nabla \times \nabla \times \mathbf{A} = \frac{\hbar q}{mc} \left(\frac{1}{2i} \left(\psi^\dagger \nabla \psi - \psi \nabla \psi^\dagger \right) - \frac{q}{\hbar c} \mathbf{A} \left| \psi \right|^2 \right) . \tag{2.6}$$

Minimization of the free energy with respect to the order field ψ is more straight forward and yields

$$\frac{\hbar^2}{2m} \left(\nabla - \frac{iq}{\hbar c} \mathbf{A} \right)^2 \psi = a \left| \psi \right|^2 \psi - \mu \psi \, . \tag{2.7}$$

This will be used to determine the structure of the flux tube.

Note that the Ginzburg-Landau equations 2.6 2.7 are invariant under the gauge transformation

$$\mathbf{A}(\mathbf{x}) \to \mathbf{A}(\mathbf{x}) + \nabla \varphi(\mathbf{x}) \, , \tag{2.8}$$

$$\psi(\mathbf{x}) \to e^{\frac{iq}{\hbar c} \varphi(\mathbf{x})} \psi(\mathbf{x}) \, . \tag{2.9}$$

This transformation can be used to remove the phase of the order parameter.

Now that we have established the field equations we can begin to apply them. The first will be a demonstration of the Meissner effect and a derivation of the London penetration depth.

2.2 The Meissner Effect

The Meissner effect follows from equation 2.6 quite nicely. Consider it in cartesian coordinates for now, where a superconducting state exists for $x > 0$ and a normal state for $x < 0$. We will use the polar decomposition of ψ as an Ansatz;

$$\psi\left(x\right) = \sqrt{\frac{\mu}{a}}\rho\left(x\right)e^{i\varphi(x)}\,, \tag{2.10}$$

where $\rho(x) = [0, 1]$ gives the fraction of the field which has condensed, 0 being a normal state and 1 being totally superconducting. The states in between are called mixed states. Substituting this Ansatz into equation 2.6 yields

$$\nabla \times \nabla \times \mathbf{A}(x) = \frac{\hbar q}{mc}\nabla\varphi\left(x\right)\frac{\mu}{a}\rho(x) - \frac{4\pi q^2}{mc^2}\mathbf{A}(x)\frac{\mu}{a}\rho(x)\,. \tag{2.11}$$

Let's assume we are looking in a region of the superconductor without many disturbances. This is the same as setting $\rho(x) = 1$ and we get

$$\nabla \times \nabla \times \mathbf{A}(x) = \frac{\hbar q\mu}{mca}\nabla\varphi(x) - \frac{4\pi q^2\mu}{mc^2 a}\mathbf{A}(x)\,. \tag{2.12}$$

Taking the curl of both sides gives us the London equation,

$$\nabla \times \nabla \times \mathbf{B}(x) = -\lambda^2\mathbf{B}(x)\,, \tag{2.13}$$

where the quantity λ is called the penetration depth and is defined as

$$\lambda = \sqrt{\frac{mc^2}{4\pi q^2 n_0}}\,, \tag{2.14}$$

and $n_0 = \frac{\mu}{a}$ is the density of superconducting electrons. Its name comes from the interpretation of the solution of the London equation, $B(x) = e^{-\frac{x}{\lambda}}$, which says that the magnetic field will decay after a characteristic length λ past the surface of the superconductor. It is also instructive to note that $q = 2e$ and $m = 2m_e$. This is consistent with the picture that Cooper pairs are responsible for the condensate. Having made sure that the theory contains the fundamental results of the London equations we can now see what new phenomena the theory predicts.

2.3 Vortex Lines

The vortex line solution comes from solving the equations of motion in cylindrical coordinates and was first discovered by Abrikosov [1]. A vortex is a cylindrically symmetric line defect which exists in an otherwise undisturbed superconductor. It is similar to the fluid vortices that are formed when water goes down a drain. In a superconductor the electrons rotate around a core where the density of superconducting electrons drops to zero. We will first investigate how this structure leads to the quantization of magnetic flux.

2.3.1 Flux Quantization

An indication that somthing interesting is happening comes form our definition of the current $\mathbf{j}$ given by 2.4. If we solve equation 2.4 for the vector potential we get

$$\mathbf{A} = -\frac{\hbar c}{q}\frac{\mathbf{j}}{|\psi|^2} + \frac{\hbar c}{q}\frac{1}{2i}\frac{1}{|\psi|^2}\left(\psi^\dagger\nabla\psi - \psi\nabla\psi^\dagger\right) . \tag{2.15}$$

In cylindrical coordinates, equation 2.7 incidates that as $r \to 0$, $\rho \to 0$. If the superconductor is not in an electric potential then current can only be produced by disturbances in the superconductor. Far away from $r = 0$, the superconductor is in an undisturbed state so $\mathbf{j} = 0$ and

$$\mathbf{A} = \frac{\hbar c}{q}\frac{1}{2i}\frac{1}{|\psi|^2}\left(\psi^\dagger\nabla\psi - \psi\nabla\psi^\dagger\right) . \tag{2.16}$$

Substituting the Ansatz for ψ gives

$$\mathbf{A} = \frac{\hbar c}{q}\nabla\varphi\left(\mathbf{x}\right) . \tag{2.17}$$

Because the phase of the wave function is single valued integrating on a closed contour around the vortex leads to a quantization condition

$$\oint \mathbf{A}\cdot d\mathbf{l} = \frac{\hbar c}{q}\oint \nabla\varphi\left(\mathbf{x}\right)\cdot d\mathbf{l} = \frac{\hbar c}{q}2\pi n\,, \tag{2.18}$$

where n is an integer. We can use Stokes theorem to see that the contour integral of $\mathbf{A}$ is also the flux through the surface,

$$\oint \mathbf{A}\cdot d\mathbf{l} = \int \nabla\times\mathbf{A}\cdot d\mathbf{S} = \int \mathbf{B}\cdot d\mathbf{S} = \Phi\,. \tag{2.19}$$

Equating the two expressions indicates that the flux is quantized with quantum number n and that

$$\Phi = n\Phi_o \text{ with } \Phi_o = \frac{2\pi\hbar c}{q}\,. \tag{2.20}$$

The integer n is called the winding number and is an indication of the strength of the vortex. It can be shown energetically that a single vortex of winding $n = N$ will decay into N vorticies each with a winding number of $n = 1$ [26]. Also notice that the flux away from the vortex is independent of the radius of the loop we integrate around. These will be considerations in choosing the Ansatz in the next section.

Right now it is unclear where the flux actually is. As it has been derived it looks like it penetrates the entire plane. To find where this might be localized we have to solve the other equation of motion governing the density of the superconductor.

2.3.2 The Structure of the Vortex

The problem with solving the field equations 2.6 2.7 in cylindrical coordinates is that they are coupled, non-linear differential equations. We are looking for defects so we will no longer assume that $\psi(x)$ is constant. To make it easier to decouple and linearize these equations it is convenient to define

$$\psi = \sqrt{\frac{\mu}{a}}\rho\left(r\right)e^{i\phi}\,, \tag{2.21}$$

$$\mathbf{A} = \frac{\hbar q}{c}\frac{a\left(r\right)}{r}\;\hat{\boldsymbol{\phi}}\,, \tag{2.22}$$

where $\rho\left(r\right),\ a\left(r\right) \to 1$ as $r \to \infty$ and $\rho\left(r\right),\ a\left(r\right) \to 0$ as $r \to 0$ and r and ϕ the cylindrical coordinates. The phase of $\psi(x)$ is chosen to mimic a vortex with winding number $n = 1$.

We can further define

$$\rho\left(r\right) = 1 + \sigma\left(r\right)\,, \tag{2.23}$$

$$a\left(r\right) = 1 + r\alpha\left(r\right)\,, \tag{2.24}$$

such that $\sigma\left(r\right),\ \alpha\left(r\right) \to 0$ as $r \to \infty$. Let's start by subsituting 2.21 and 2.22 into equation 2.6.

$$\nabla\times\nabla\times\left(\frac{a(r)}{r}\;\hat{\boldsymbol{\phi}}\right) = \frac{4\pi\hbar q^2}{mc}\frac{\xi\mu}{a}\left(\frac{1}{r} - \frac{a(r)}{r}\right)\rho(r)^2\;\hat{\boldsymbol{\phi}} \tag{2.25}$$

Writing the cross products in cylindrical coordinates and using 2.23 and 2.24 we get

$$\frac{\partial^2\alpha}{\partial r^2} + \frac{1}{r}\frac{\partial\alpha}{\partial r} - \frac{1}{r^2} - \frac{\alpha}{r^2} = \frac{\hbar\pi\mu q^2}{mca}\alpha(1+\sigma)^2\,. \tag{2.26}$$

We can linearize this equation by taking $r \to \infty$. We keep only the terms linear in α and σ and the lone $\frac{1}{r^2}$ goes to zero. This leaves us with a modified Bessel equation of the first order,

$$\frac{\partial^2\alpha}{\partial r^2} + \frac{1}{r}\frac{\partial\alpha}{\partial r} - \left(\frac{1}{r^2} + \frac{1}{\lambda^2}\right)\alpha = 0\,, \tag{2.27}$$

where λ is the London peretration depth we derived earlier. We want a solution that goes to zero as $r \to \infty$ so we choose the solution to be a modified Bessel function of the second kind, $\alpha = \frac{1}{\lambda}K_1\left(\frac{r}{\lambda}\right)$. Going back through all the substitutions find that the vector potential is

$$\mathbf{A} = \frac{\hbar c}{qr}\left[1 - \frac{rc_\phi}{\lambda}K_1\left(\frac{r}{\lambda}\right)\right]\;\hat{\boldsymbol{\phi}}\,, \tag{2.28}$$

where c_ϕ is just a constant from solving the differential equation. This describes the Meissner effect in cylindrical coordinates. As we move from the core of the vortex into the superconducting material the magnetic field decays. We will do a gauge transform on this later to put in a more recognisable form.

We will now look at the structure of $|\psi|^2$ as a function of the radius. We want solutions such that the superfluid density starts at zero in the core and goes to $\frac{a}{\mu}$ at infinity. We start by substituting 2.21 and 2.24 into 2.7,

$$\frac{\hbar^2}{2m}\sqrt{\frac{\mu}{a}}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \rho}{\partial r}-\frac{\rho}{r^2}-\frac{a^2\rho}{r^2}+\frac{2a\rho}{r^2}\right)=\left(-\rho+\rho^3\right)\frac{\mu^{\frac{3}{2}}}{a^{\frac{1}{2}}}\,. \tag{2.29}$$

Substituting in 2.23 and 2.24 and linearizing yields

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \sigma}{\partial r}=\frac{4m\mu}{\hbar^2}\sigma\,. \tag{2.30}$$

This is a modified bessel equation of the zeroth order, $K_0\left(\frac{\sqrt{2}}{\xi}r\right)$. Going back through all the substitutions we find the asymtotic solution to the order field to be

$$\psi=\sqrt{\frac{\mu}{a}}\left[1-c_\sigma K_0\left(\frac{\sqrt{2}}{\xi}r\right)\right]e^{i\phi}\,, \tag{2.31}$$

where c_σ is a constant from solving the DE. The quantity ξ is called the coherence length and is defined as

$$\xi=\sqrt{\frac{\hbar^2}{2m\mu}}\,. \tag{2.32}$$

It gives a length scale for the change in density $|\psi|^2$ from the non-superconducting core at $r=0$ to the undisturbed superconductor $r=\mathcal{O}(\xi)$. The length scale is a measure of the size of the vortex. Later we will make the approximation that superconductivity is completly destroyed inside the radius ξ. This aids in the evaluation of integrals.

It is useful to simplify these solutions using a guage trasformation which removes the phase of the order field,

$$\psi\left(\mathbf{x}\right)\rightarrow\psi e^{-i\phi}\,. \tag{2.33}$$

$$\mathbf{A}\left(\mathbf{x}\right)\rightarrow\mathbf{A}\left(\mathbf{x}\right)-\frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{\hbar c}{q}\phi\right)\ \hat{\boldsymbol{\phi}}\,, \tag{2.34}$$

The solutions to the field equations simplify to,

$$\psi=\sqrt{\frac{\mu}{a}}\left[1-c_\sigma K_0\left(\frac{\sqrt{2}}{\xi}r\right)\right]\,, \tag{2.35}$$

$$\mathbf{A}=\frac{\hbar c}{q\lambda}c_\phi K_1\left(\frac{r}{\lambda}\right)\ \hat{\boldsymbol{\phi}}\,. \tag{2.36}$$

This form will be useful later when we calculate the interaction between two vortices.

We can start to understand what is happening in a vortex. It is a disturbance which has a non-superconducting core of radius ξ that carries quanta of magnetic flux. Because the superconductor displays the Meissner effect this flux can only be carried along the core of the vortex, where superconductivity is destroyed. Essentially the vortex is a tube of magnetic flux allowing a magnetic field to penetrate the superconductor.

2.4 The Energy of a Vortex

Having solved the field equations for $\psi(x)$ and $\mathbf{A}(x)$ it is now possible to get a rough estimate of the energy of a vortex. If we take the field equation 2.7 and substitute it into equation 1 we get an expression for the free energy,

$$E = \int d^2x \left\{ \frac{\mu^2}{a}\rho^4 + \frac{(\nabla \times \mathbf{A})^2}{8\pi} \right\} . \tag{2.37}$$

We know that zeroth order Bessel functions have the property that

$$\frac{dK_0(\alpha r)}{dr} = -\alpha K_1(\alpha r) \tag{2.38}$$

and, in cylindrical coordinates, are a Green's function for

$$(\nabla^2 - \alpha^2)K_0(\alpha r) = -2\pi\delta^2(\mathbf{r}) . \tag{2.39}$$

Using these on 2.28, when $r \neq 0$, we get

$$\nabla \times \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}\frac{r}{\lambda}K_1(\frac{r}{\lambda}) = -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}K_0(\mu r) = -\mu^2 K_0(\frac{r}{\lambda}) . \tag{2.40}$$

It can be shown that at large r the leading order of ρ^4 is $\frac{\xi^2}{2r^2}$ [16]. Using all of this the energy becomes

$$E = \int_0^{2\pi}\int_0^{\infty} r dr d\phi \left\{ \frac{\mu^2}{a}\frac{\xi^2}{2r^2} + \frac{\left(\frac{\hbar c}{q\lambda^2}K_0\left(\frac{r}{\lambda}\right)\right)^2}{8\pi} \right\} . \tag{2.41}$$

The first term is easily integrated, but is unbounded at its limits. Instead of $r = \infty$ we use a cutoff $r = \Lambda$ which is the size of the container holding the superconductor. We remove the singularity at $r = 0$ by neglecting the core of the vortex $r < \xi$. The second term is evaluated by using $\int_0^\infty rK_0^2(r)\,dr = \frac{1}{2}$. The energy per unit length of the vortex becomes

$$E = \frac{\pi\hbar^2}{m}\frac{\mu}{a}\log\left(\frac{\Lambda}{\xi}\right) + \frac{1}{8}\left(\frac{\hbar c}{q\lambda}\right)^2 . \tag{2.42}$$

The fact that a cutoff is required indicates that a vortex can only exist in a container of finite size. Now that we have calculated the energy for a single vortex it will be interesting to look at two vorticies and the interaction energy between them.

2.5 Interaction Energy Between Two Vortices

The philosophy behind calculating the interaction between vortices is to find the energy of the entire system and then subtract off the energy of the individual vorticies as originally outlined by Kramer [17]. The technique we will use was introduced by Speight [37] and used by Buckley et al. [15] and MacKenzie et al [32] to calculate vortex interactions in models with two order parameters[1]. The same philosophy is used but the actual calculation becomes much less cumbersome. We will reduce the theory to a non-interacting, linear one and then model the vorticies as point sources. The interaction energy is then calculated from this linear theory.

To aid in the calculation it is useful to remove the phase in ψ as was done earlier 2.35 2.36 and write ϕ as $\sqrt{\frac{\mu}{a}}(1-\sigma)$. To linearize the theory we expand in σ and $\mathbf{A}$ and keep only quadratic terms to get

$$E_{\text{free}} = \int d^2x \left\{ \frac{\mu}{a}\frac{\hbar^2}{2m}(\nabla\sigma)^2 + \frac{1}{8\pi}\left((\nabla \times \mathbf{A})^2 + \frac{\mathbf{A}^2}{\lambda^2} \right) + 2\frac{\mu^2}{a}\sigma^2 \right\} . \tag{2.43}$$

The source terms are

$$E_{\text{source}} = \int d^2x \left\{ \tau\sigma + \mathbf{j}\cdot\mathbf{A} \right\} , \tag{2.44}$$

where τ and $\mathbf{j}$ are the sources for the fields σ and $\mathbf{A}$. Minimizing $E_{\text{free}} + E_{\text{total}}$ we get the equations of motion,

$$\left(\nabla^2 - \frac{2}{\xi^2}\right)\sigma = \frac{m}{\hbar^2}\frac{a}{\mu}\tau , \tag{2.45}$$

$$\left(\nabla^2 - \frac{1}{\lambda^2}\right)\mathbf{A} = 4\pi\mathbf{j} . \tag{2.46}$$

We want to solve for the sources $\mathbf{j}$ and τ such that they have the same asymptotic solutions we obtained earlier in 2.28 and 2.31. Using 2.39 and the derivative of 2.39 we can solve for the sources,

$$\tau = -\frac{\hbar^2}{m}\frac{\mu}{a}2\pi\delta^2(\mathbf{x}) , \tag{2.47}$$

$$\mathbf{j} = \frac{\hbar c}{2q}\frac{\partial\delta^2(\mathbf{x})}{\partial x}\ \hat{\boldsymbol{\phi}} . \tag{2.48}$$

The interaction energy is found by substituting $\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2$, $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$, $\tau = \tau_1 + \tau_2$ and $\sigma = \sigma_1 + \sigma_2$ into the total energy $E = E_{\text{free}} + E_{\text{source}}$ and subtracting of the energies of the vortices, leaving only cross terms. The subscripts 1 and 2 refer to two seperate vorticies and positions $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. The cross terms left over are interpreted as the interaction energy,

$$E_{\text{interaction}} = \int d^2x \left\{ \tau_1\sigma_2 + \mathbf{j}_1\cdot\mathbf{A}_2 \right\} . \tag{2.49}$$

[1] A simple example of this technique is outlined in Appendix A.

Using 2.22, 2.23, 2.47 and 2.48 the interaction energy can be written as

$$\begin{aligned} E_{\text{interaction}} &= \int d^2x \left\{ -\frac{\hbar^2}{m}\frac{\mu}{a} 2\pi\delta^2(\mathbf{x}-\mathbf{x}_1) K_0\left(\frac{\sqrt{2}}{\xi}(\mathbf{x}-\mathbf{x}_2)\right) \right. \\ &\left. + \frac{\hbar c}{2q}\frac{\partial\delta^2(\mathbf{x}-\mathbf{x}_1)}{\partial x}\frac{\hbar c}{q\lambda} K_1\left(\frac{\mathbf{x}-\mathbf{x}_2}{\lambda}\right)\right\}, \\ &= \left(\frac{\hbar c}{q\lambda}\right)^2 \frac{1}{2} K_0\left(\frac{d}{\lambda}\right) - \frac{2\pi\hbar^2\mu}{ma} K_0\left(\frac{\sqrt{2}d}{\xi}\right), \\ &= \frac{2\pi\hbar^2\mu}{ma}\left(K_0\left(\frac{d}{\lambda}\right) - K_0\left(\frac{\sqrt{2}d}{\xi}\right)\right), \end{aligned} \tag{2.50}$$

where $d = |\mathbf{x}_1 - \mathbf{x}_2|$. Integration of the second term is subtle and the steps are clearly outlined in equations 3.28.

There are two terms working to oppose each other. The first term is a repulsive force similar to the force between two wires with currents in opposite directions. The current in this case is caused by the electrons rotating around the vortex. Two vortices placed side by side will have currents running in the opposite direction and be repelled. We can see from equation 2.48 that the current is in the $\hat{\boldsymbol{\phi}}$ direction around the vortex. The second term is an attractive force caused by the superconductor prefering to be in a state with no defects and attemping to restore order by making only one vortex. When the first term is larger $E_{\text{interaction}}$ is positive and the vorticies repel. When the second term is larger the vortices attract. What governs this is the relative size of λ and $\frac{\sqrt{2}}{\xi}$. Because $K_0(x)$ is monotonically decreasing for all real x the vortices will repel when

$$\frac{d}{\lambda} < \frac{\sqrt{2}d}{\xi}\,. \tag{2.51}$$

Rearranging yields the famous Ginzburg-Landau parameter,

$$\kappa > \frac{1}{\sqrt{2}} \quad \text{where} \quad \kappa = \frac{\lambda}{\xi}\,. \tag{2.52}$$

This dimensionless quantity is used to determine whether a superconductor is type I or type II. A type II superconductor is one which allows partial penetration of a magnetic field. A type I superconductor is one which fully displays the Meissner effect. If $\kappa > \frac{1}{\sqrt{2}}$ then vortices repel from each other and they will form a triangular lattice [38] [39], each vortex carrying a quanta of flux Φ_0. This accounts for the partial penetration of the magnetic field exhibited by type II superconductors. If $\kappa < \frac{1}{\sqrt{2}}$ then all the vortices attract each other and collapse. The superconductor now has no mechanism to carry flux and exhibits the Meissner effect, behaving like a type I superconductor.

2.6 Critical Magnetic Fields

Type I and type II supercondunductors have another distinguishing feature, the magnetic fields at which the Meissner effect is destroyed. A type I superconductor will display the Meissner effect until a critical external field B_{c_I} destroys the superconducting state.

A type II superconductor will display the Meissner effect until a critical field $B_{c_{II}}$ when vortices start to form and allow part of the field to penetrate it. Increasing the magnetic field strength further will create more and more vorticies until there are so many that superconductivity is destroyed.

Let us first consider a type I superconductor. The density ψ is uniform and there is no magnetic field inside so equation 1 becomes

$$E_{\text{condensate}} = V\left(\frac{a}{2}|\psi|^4 - \mu|\psi|^2\right) , \tag{2.53}$$

where V is the volume of the superconductor. This has minima at $|\psi|^2 = \frac{\mu}{a}$. Applying an external magnetic field changes the energy by $-\frac{B^2}{8\pi}$. If we set $E_{\text{condensate}} = 0$ the condenstate has been destroyed and we get a critical magnetic field

$$B_{c_I} = \sqrt{\frac{4\pi\mu^2}{a}} . \tag{2.54}$$

Now consider a type II superconductor. There are both energy gradients and magnetic fields inside the superconductor. We use the energy of a vortex we calculated earlier and this time the magnetic field inside the vortex B_{int} couples with the external field B_{ext} through the interaction term

$$E_{\text{vortex}} = \frac{\pi\hbar^2}{m}\frac{\mu}{a}\log\left(\frac{\Lambda}{\xi}\right) + \frac{1}{8}\left(\frac{\hbar c}{q\lambda}\right)^2 - \int d^2x\frac{\mathbf{B}_{\text{int}}\cdot\mathbf{B}_{ext}}{8\pi} . \tag{2.55}$$

If the external magnetic field is in the $\hat{\boldsymbol{z}}$ direction then the last term is just a statement of flux quantization and can be simplified, $\int_0^\infty rdr2\pi B_{\text{int}} = 2\pi\frac{\hbar c}{q}$. The energy $E = 0$ is when a vortex will first form inside the superconductor so,

$$B_{c_{II}} = \frac{4\pi\hbar q\mu}{mca}\left(\frac{1}{4} + \frac{1}{\pi}\log\left(\frac{\Lambda}{\xi}\right)\right) . \tag{2.56}$$

Comaparing the two critical fields we see that $B_{c_{II}}$ is much smaller than B_{c_I}. This is expected because a type II superconductor only has to let one quantum of flux through at $B_{c_{II}}$, where B_{c_I} has the energy to destroy the entire state.

There will be a critical magnetic field $B^{(2)}_{c_{II}}$ in a type II superconductor when so many vortices have penetrated the bulk that superconductivity is destroyed. Abrikosov noticed that just before this happens the order parameter is very small. This means that only the non-linear terms of the equation of motion 2.7 need to be considered and we are left with

$$-\frac{\hbar^2}{2m}\left(\nabla - \frac{iq}{\hbar c}\mathbf{A}\right)^2\psi = \mu\psi . \tag{2.57}$$

If v is interpreted as the energy level then this is just the Schrödinger equation for a particle in a magnetic field. The minimum energy of a particle in a magnetic field is just $E_0 = \frac{\hbar q B}{2mc}$. Then, by analogy, the particle only exists if

$$\mu > \frac{\hbar q B}{2mc} \rightarrow B_{c_{II}}^{(2)} < \frac{2mc\mu}{q\hbar} \ . \tag{2.58}$$

A magnetic field above $B_{c_{II}}^{(2)}$ will destroy superconductivity in a type II superconductor.

Chapter 3

Superfluid Vortices and Sheets

Ginzburg-Landau theory can be used to describe the energy of superfluid vortices if the limit of the charge $q \to 0$ is taken. This effectively shuts off the magnetic field leaving only the terms from the order parameter. The free energy becomes

$$E = \int d^2x \left\{ \frac{\hbar^2}{2m} |\nabla\psi|^2 - \mu |\psi|^2 + \frac{a}{2} |\psi|^4 \right\} . \tag{3.1}$$

Vortex structures in fluids arise when the vessel containing the fluid is rotated. With a normal fluid, one with viscosity, friction transfers the energy from the rotating container to the fluid causing it to rotate. The particles then transfer this energy throughout the fluid until the whole volume is rotating as a whole and is indistinguishable from solid body rotation.

For a superfluid[1], where the viscocity is zero, this is unlikely. The energy required to keep the atoms rigid is extremly high, so there must be another solution. The equation which describes the minimum energy in the rotating frame for a given Ω [20] [19] is

$$E_{\text{rot}} = E - \mathbf{\Omega} \cdot \mathbf{M} , \tag{3.2}$$

where E and $\mathbf{M}$ are the energy and angular momentum of the system in the fixed coordinate system and $\mathbf{\Omega}$ is the angular velocity of the container. As $\mathbf{\Omega}$ increases it becomes more and more favorable for the fluid to carry angular momentum. Eventually the superfluid must move somehow. This chapter will describe the structure of the superfluid and this movement. There are a number of good reviews on superfluid vortices but the bulk of the knowledge in this chapter was culled from Kleinert [16], Landau [21] and Feynman [8].

3.1 The Structure of a Superfluid Vortex

Here we will look at how the density of the superfluid changes when a single vortex is formed. Minimizing equation 3.1 yields the equation of motion which

[1] The consequnces of rotating a superconductor are discussed in Appendix B

will determine the structure of the vortex. This can aslo be obtained directly from the cylindrical solution to equation 2.7 with $q \to 0$,

$$\frac{\hbar^2}{2m}\nabla^2\psi = a\,|\psi|^2\,\psi - \mu\psi\,. \tag{3.3}$$

We use a polar decomposition ansatz for the wave function of the condensate,

$$\psi\,(x) = \sqrt{\frac{\mu}{a}}\rho\,(x)\,e^{iN\varphi(x)}\,, \tag{3.4}$$

where N is the winding number of the vortex, $\frac{\mu}{a}$ is identified as the number density, n_0, of the undisturbed superfluid, and ρ is a function which can take values from 0 to 1 to describe changed in the superfluid number density. Substituting this in yields

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\rho}{\partial r} - \frac{N^2\rho}{r^2} = \frac{2mv}{\hbar^2}(\rho^3 - \rho)\,. \tag{3.5}$$

We want to look at asymptotic behavior of this equation to find out what happens to the superfluid as $r \to 0$, near the core of the vortex, and how the superfluidity gets restored away from the vortex at $r \to \infty$. As $r \to 0$ equation 3.5 is dominated by the left hand side of the equation and we take the right hand side to be zero. This is similar to a Bessel differential equation in the same limit. Because we want $\rho = [0, 1]$ we choose the non-singular Bessel function of the first kind as a solution,

$$\rho = c_\rho J_N\left(\frac{\sqrt{2}}{\xi}r\right) \quad \text{where} \quad \xi = \sqrt{\frac{\hbar^2}{2m\mu}}\,, \tag{3.6}$$

where c_ρ is a constant. This shows that deep in the core of the vortex the superfluid density $\rho_\mathrm{s} = m|\phi|^2$ drops to zero. Landau described ^4He with a two fluid model where the total density of particles is the sum of the super fluid and normal phases $\rho_\mathrm{tot} = \rho_\mathrm{s} + \rho_\mathrm{n}$. This implies that in the core of the vortex there exists only normal liquid ^4He.

To get the other limit it is helpful to make the substitution $\rho = 1 + \sigma$. We assume that far away from the vortex $\rho \to 1$ which implies that as $r \to \infty$, $\sigma \to 0$. Making this substitution and then linearizing yields

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\sigma}{\partial r} - \frac{N^2\sigma}{r^2} = \frac{4m\mu}{\hbar^2}\sigma\,, \tag{3.7}$$

which is a modified Bessel differential equation. The solution which makes physical sense and has the correct asymptotics is the modified bessel function of the second kind. Substituting back gives the solution

$$\rho = 1 - c_\rho K_N\left(\frac{\sqrt{2}}{\xi}r\right)\,, \tag{3.8}$$

which gives the behavior of the superfluid at long range. Notice that as the winding number increases the order of the Bessel function increases and that

Bessel functions go to zero more slowly with increasing order. So as winding number increases the radius of the vortex increases meaning the defect is stronger.

3.2 Quantization of Circulation

Now that we have the distribution of the superfluid density we can ask what the superfluid is doing. Just as Abrikosov did for superconducting vortices, Feynman brought forward the idea of quantized superfluid vortices [7][2].

It has been shown that the ansatz 3.4 gives a reasonable structure for a vortex. In the previous chapter we found that for a superconducting vortex the magnetic flux carried in the core was quantized. This was found using a line integral about the vortex. There is a quantity in fluid dynamics called the circulation which has a similar form;

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l}\,. \tag{3.9}$$

Using the wave function we can find the velocity of the superfluid using the correspondence rule $\hat{\mathbf{v}} = \frac{\mathbf{p}}{m} = -i\frac{\hbar}{m}\nabla$. Let us consider a point in the super fluid away from the vortex where the superfluid density isn't changing. Then acting the velocity operator on the wave function gives

$$\hat{\mathbf{v}}\ \psi = -i\frac{\hbar}{m}\nabla\sqrt{n_0}\rho e^{iN\varphi(x)} = \frac{\hbar}{m}(\nabla\varphi)\psi\,, \tag{3.10}$$

or more clearly

$$\mathbf{v} = \frac{\hbar}{m}\nabla\varphi\,. \tag{3.11}$$

The velocity distribution can also be found by using the definition of the current 2.4 and substituting in the wave function,

$$\mathbf{j} = \frac{i\hbar}{2m}\left(\psi\nabla\psi^\dagger - \psi^\dagger\nabla\psi\right) \tag{3.12}$$

$$= \frac{\hbar}{m}n_0\nabla\varphi \tag{3.13}$$

This is the macroscopic current denisity of condensate particles which can be equated to $\mathbf{j} = n_0\mathbf{v}$. Then the velocity of the fluid is exactly what we obtained above. Finding the fluid velocity using the current helps unify the pictures of superfluid vortices and superconducting vortices. All one has to do is take the limit $q \to 0$ in a gauge theory to get superfluid behaviour.

[2] As a historical note, Landau was a large proponent of domains in superconductors and vortex sheets in superfluids [18]. He hated the idea of vortices in superconductors so much that he prevented Abrikosov (who was Landau's student at the time) from publishing his superconducting vortex solution. It was only after he saw that Feynman's superfluid vortex solution was correct that he let Abrikosov publish his solution for superconductors

A velocity field described by the gradient of a function is called a potential flow. The first thing to note from this is that the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ of a potential flow is

$$\nabla \times \mathbf{v} = \frac{\hbar}{m} \nabla \times (\nabla \varphi) = 0 \, . \tag{3.14}$$

When the vorticity is zero the fluid is called irrotational because, in a simply connected region, all closed paths can be shrunk to a point and the fluid velocity is zero everywhere. In a multiply connected region, such as when a vortex is present, the circulation about any closed curve that can't be shrunk to a single point does not need to vanish. If $\varphi(x)$ is set to be the the azimuthal angle, ϕ, which increases by 2π as of going around the hole, the fact that the wave function is single valued will lead to the circulation being quantized,

$$\oint \mathbf{v} \cdot d\mathbf{l} = \frac{\hbar}{m} \oint \nabla \varphi(\mathbf{x}) \cdot d\mathbf{l} = 2\pi N \frac{\hbar}{m} \, , \tag{3.15}$$

where a quanta of qirculation is given as

$$\Gamma_0 = \frac{2\pi\hbar}{m} \, . \tag{3.16}$$

Equation 3.15 also gives us a proof that no vortex line can end in the middle of the superfluid. First assume that a vortex line could end in the middle of the superfluid. If we calculate the circulation around it we find that

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} = N\Gamma_0 \, , \tag{3.17}$$

But the equation for Γ can be rewritten using Stokes theorem and because the vortex ends in the middle of the superfluid there is a way to draw the surface S such that the vortex does not pass though it. Consequently there is no flux through the surface which means that the integral is zero,

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} = \int \nabla \times \mathbf{v} \; d\mathbf{S} = 0 \, , \tag{3.18}$$

which leads to a contradiction with 3.17. Thus a vortex filament must either go to infinity or stop at the boundary of the superfluid.

3.3 Energy of a Single Vortex

The energy of a superfluid vortex can be found by considering the change in kinetic energy of the superfluid upon the appearance of a vortex. It is simply

$$\Delta E = \int \frac{1}{2} \rho_s v_s^2 dV \, . \tag{3.19}$$

In order to evaluate this integral we need the distribution of the superfluid velocity. Luckily, the circulation gives us a way of finding this. Consider the simplest case of a vortex in a superfluid in which the filament is at the origin lined up with the axis of the container's rotation. In this case the streamlines of the velocity field are circles centred on the vortex in a plane perpendicular to its axis. The circulation in this case is just $2\pi r v_s$ so that

$$2\pi r = N\Gamma_0 \Longrightarrow v_s = \frac{N\Gamma_0}{r} \,. \tag{3.20}$$

Note how this potential rotation velocity distribution differs from that of a solid body. Here the velocity decreases as we move away from the centre where it would increase with solid body rotation.

We can now substitute this velocity into equation 3.19. To do the integral it is helpful to approximate the vortex as having no superfluid at all inside the vortex coherence length ξ. If the superfluid is in a container or radius R then

$$\Delta E = 2\pi L \rho_s \int_{xi}^{R} \frac{N\Gamma_0}{r} r dr = L\pi\rho_s N^2 \Gamma_0^2 \ln\left(\frac{R}{\xi}\right) \,. \tag{3.21}$$

From this it is clear to see that the energy is minimized when the winding number is $N = 1$. The energy also scales as N^2 which means that if two vortices are present it is more favorable for there to be two vortices with $N = 1$ than one large vorex with $N = 2$. This means that it is more stable for a large vortex to break into smaller ones and implies that the inter-vortex force is repulsive, which is different from the order parameter term in gauge vortices 2.50. The interaction between two vortices will be explored in detail in a later section. We also see that a vortex which is not contained has infinite energy and thus can't exist.

3.4 Interaction Between Two Vortices

The absence of a gauge field changes the way the order field interacts with itself. In the gauged field theory the order field worked to attract the superconducting vortices while the gauge field is what pushed them apart. Energy considerations for superfluid vortices show that the energy is least when the winding number N is the smallest. This means a large vortex with $N > 1$ tends to decay into a number of small vortices with $N = 1$.

The first check will be to take the $q \to 0$ limit of the Ginzburg-Landau parameter derived earlier. If $\kappa > \frac{1}{\sqrt{2}}$ then we will be in type II behavior and the vortices should repel. Taking the limit we see that

$$\lim_{q\to 0} \kappa = \lim_{q\to 0} \frac{\lambda}{\xi} = \lim_{q\to 0} \sqrt{\frac{mc^2}{4\pi q^2 n_0}\frac{2m\mu}{\hbar^2}} \sim \lim_{q\to 0} \frac{1}{q} = \infty \,. \tag{3.22}$$

Comparing with 2.52 we see that the $q \to 0$ limit of the Ginzburg-Landau parameter shows that the ungauged votices should repel.

A better check is to actually calculate the interaction energy in a calculation similar to what was done for superconducting vortices. As in the previous chapter we will use the method of Speight [37]. Linearizing the theory to be quadratic in σ yields

$$E_{\text{free}} = \int d^2x \left\{ \frac{\mu}{a} \frac{\hbar^2}{2m} (\nabla\sigma)^2 + 2\frac{\mu^2}{a}\sigma^2 \right\} . \tag{3.23}$$

Linearizing removes the structure of the vortex so we add a source term to model it;

$$E_{\text{source}} = \int d^2x \tau\sigma \, . \tag{3.24}$$

The equation of motion for $E_{\text{free}} + E_{\text{source}}$ is

$$\left(\nabla^2 - \frac{2}{\xi^2}\right)\sigma = \frac{m}{\hbar^2}\frac{a}{\mu}\tau \, . \tag{3.25}$$

Using the Green's function to solve for τ gives us

$$\tau = \frac{\hbar^2}{m}\frac{\mu}{a} 2\pi \frac{\partial\delta^2(\mathbf{x})}{\partial x} \, . \tag{3.26}$$

Now that we've found a delta fuction source which mimics the asymptotics of the real solutions we can calculate the interaction between two vortices. The interaction energy is found by substituting $\tau = \tau_1 + \tau_2$ and $\sigma = \sigma_1 + \sigma_2$ into the total energy $E = E_{\text{free}} + E_{\text{source}}$ and subtracting of the energies of the vortices, leaving only cross terms. The subscripts 1 and 2 refer to two seperate vortices and positions $\mathbf{a}_1$ and $\mathbf{a}_2$ respectivly. The cross terms left over are interpreted as the interaction energy,

$$E_{\text{int}} = \int d^2x \tau_1 \sigma_2 \, . \tag{3.27}$$

Substituting gives us

$$
\begin{aligned}
E_{\text{int}} &= \int d^2x \frac{\hbar^2}{m}\frac{\mu}{a}2\pi\frac{\partial\delta^2(\mathbf{x}-\mathbf{x}_1)}{\partial r}K_1\left(\frac{\sqrt{2}}{\xi}(\mathbf{x}-\mathbf{x}_2)\right)\,, && (3.28)\\
&= \int rdrd\theta\frac{2\pi\hbar^2}{m}\frac{\mu}{a}\frac{\partial}{\partial r}\left(\frac{\delta(r-r_1)}{r_1}\delta(\theta)\right)K_1\left(\frac{\sqrt{2}}{\xi}(r-r_2)\right)\,, && (3.29)\\
&= -\int dr\frac{2\pi\hbar^2}{m}\frac{\mu}{a}\delta(r-r_1)\frac{1}{r_1}\frac{\partial}{\partial r}\left(rK_1\left(\frac{\sqrt{2}}{\xi}(r-r_2)\right)\right)\,, && (3.30)\\
&= -\frac{2\pi\hbar^2}{m}\frac{\mu}{a}\frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1K_1\left(\frac{\sqrt{2}}{\xi}(r_1-r_2)\right)\right)\,, && (3.31)\\
&= \frac{2\pi\hbar^2}{m}\frac{\mu}{a}\frac{\xi}{\sqrt{2}}\frac{1}{r_1}\frac{\partial}{\partial r_1}r_1\frac{\partial}{\partial r_1}K_0\left(\frac{\sqrt{2}}{\xi}(r_1-r_2)\right)\,, && (3.32)\\
&= \frac{2\pi\hbar^2}{m}\frac{\mu}{a}\frac{\xi}{\sqrt{2}}\frac{2}{\xi^2}K_0\left(\frac{\sqrt{2}}{\xi}d\right)\,, && (3.33)\\
&= \left(\frac{\hbar c}{q\lambda}\right)^2\frac{1}{\sqrt{2}\xi}K_0\left(\frac{\sqrt{2}}{\xi}d\right)\,, && (3.34)\\
& && (3.35)
\end{aligned}
$$

where $d = |\mathbf{r}_1 - \mathbf{r}_2|$ and represents the distance between the vortices. In the absence of a gauge field the intereaction energy is positive and the order field is now responsible for a repulsive force. Superfluid vortices repelling each other is consistent with the previous energy considerations involving the winding number and the $q \to 0$ limit of the Ginzburg-Landau parameter.

3.5 Critical Ω and the Areal Density of Vortices

The quantization of circulation has led to a quantization of the energy of a vortex. Because the angular velocity, Ω, is resposible for putting energy into the system, which is quantized, it stands to reason that there is a critical Ω_c at which a votex line will first appear in the superfluid. This specifically occurs when Ω is such that the creation of a vortex lowers the energy in the rotating frame, specifically when

$$\Delta E_{\text{rot}} = \Delta E - M\Omega_c = 0\,. \tag{3.36}$$

The angular momentum M can be calculated classically from

$$M = \int \rho_s v_s^2 r dV = \rho_s \Gamma_0 \int dV = L\pi R^2 \rho \Gamma_0\,. \tag{3.37}$$

Substituting 3.37 and 3.21 into 3.36 and solving for Ω_c yields

$$L\pi\rho_s N^2\Gamma_0^2 \ln\left(\frac{R}{\xi}\right) = L\pi R^2\rho\Gamma_0\Omega_c \Longrightarrow \Omega_c = \frac{\Gamma_0}{R^2}\ln\left(\frac{R}{\xi}\right). \qquad (3.38)$$

Vortex lines exist for $\Omega > \Omega_c$. When $\Omega \gg \Omega_c$ a large number of vortex lines form. As shown earlier it is favorable for the vortices to repel each other and form a lattice in the container similar to that of superconducting vortices.

It is apparent that a greater Ω will create more and more vortices. It is necessary to minimize equation 3.2 for a certain number of vortices. Even with quantized circulation and potential flow this still occurs when the motion mimics solid body rotation. The vorticity of a solid body of radius R is $\nabla \times \mathbf{v} = 2\mathbf{\Omega}$ which means the circulation by Stokes theorem is

$$\Gamma_{\text{solid}} = \int \nabla \times \mathbf{v} d\mathbf{S} = 2\Omega \int dS = 2\pi R^2 \Omega\,. \qquad (3.39)$$

The circulation of a lattice of vortices can also be calculated by multiplying the number of vortices enclosed by the circulation, N_{v}, by their circulation Γ_0. Equating the two gives the Onsager-Feyman formula for the areal density of vortices [7],

$$2\pi R^2\omega = N_{\text{v}}\Gamma_0 \Longrightarrow n_{\text{v}} = \frac{2\Omega}{\Gamma_0}\,, \qquad (3.40)$$

where $n_{\text{v}} = \frac{N_{\text{v}}}{\pi R^2}$ is the vortex density for which the system mimics the angular momentum of a rotating solid body.

3.6 Energy of an Array of Vortices

We are now in a position to calculate the total energy of a superfluid in a rotating bucket which has a large number of vortices present. This calculation was first presented by Hall [9] [10] [11] as proof that Feynman's vortex line solution was more favorable for ^4He than Landau's vortex sheets.

Consider a cylindrical container of radius R rotating about its axis with angular velocity Ω_0. Assuming that Ω_0 is fast enough to create a number of vortex lines N_{v} we want to minimize the free energy with respect to N_{v}. The vortices are arranged such that the relation

$$r^2 = \frac{N_{\text{v}}\Gamma_0}{2\pi\Omega} \qquad (3.41)$$

holds, where Ω indicates the circulation required for the superfluid to simulate solid body rotation, not the angular velocity of the container.

If the fluid only creates vortices inside the radius r, the energy of the system E and the angular momentum M carried by it are

$$E = \frac{\rho_s\pi}{4}r^4\Omega^2 + \frac{\rho_s N_{\text{v}}\Gamma_0^2}{4\pi}\ln\left(\frac{\Lambda}{\xi}\right) + \frac{\rho_s N_{\text{v}}^2\Gamma_0^2}{4\pi}\ln\left(\frac{R}{r}\right), \qquad (3.42)$$

$$M = \frac{\rho_s \pi}{2} r^4 \Omega + \frac{\rho_s \pi \Gamma_0}{4} r^2 + \frac{\rho_s \pi N_\mathrm{v} \Gamma_0}{2} \left(R^2 - r^2\right)^2 , \tag{3.43}$$

where b is a distance the order of the vortex spacing which, from equation 3.40, is $\Lambda \propto 1/\Omega^{\frac{1}{2}}$. The first term in these equations is assosiated with the solid body rotation inside of the radius r which the vortex structure mimics. This takes into account the macroscopic movement of the system. The second term takes into account the effect of the individual vortices. The energy from the vortices is the number of vortices multiplied by the energy of a single vortex 3.21 and the angular momentum of many vortices can be found using equation 3.37. The third term comes from the superfluid outside the radius r where there are no vortices present.

Substituting 3.41, 3.42 and 3.43 into $E' = E - M\Omega_0$ gives the free energy

$$\begin{aligned} E' &= \frac{\rho_s \pi}{4} \left(\frac{N_\mathrm{v}\Gamma_0}{2\pi}\right)^2 + \frac{\rho_s N_\mathrm{v} \Gamma_0^2}{4\pi} \ln\left(\frac{1}{\Omega^{\frac{1}{2}}\xi}\right) , \\ &+ \frac{\rho_s N_\mathrm{v}^2 \Gamma_0^2}{4\pi} \ln\sqrt{\frac{2\pi R^2 \Omega}{N_\mathrm{v}\Gamma_0}} - \frac{\rho_s \pi}{2} \left(\frac{N_\mathrm{v}\Gamma_0}{2\pi}\right)^2 \frac{\Omega_0}{\Omega} , \\ &- \frac{\rho_s \pi \Gamma_0 \Omega_0}{4} \left(\frac{N_\mathrm{v}\Gamma_0}{2\pi\Omega}\right) - \frac{\rho_s \pi N_\mathrm{v} \Gamma_0 \Omega_0}{2} \left(R^2 - \frac{N_\mathrm{v}\Gamma_0}{2\pi\Omega}\right)^2 . \end{aligned} \tag{3.44}$$

Minimizing this with respect to Ω gives the equilibrium value for the angular velocity required for the superfluid to mimic solid body rotation,

$$\frac{\partial F'}{\partial \Omega} = \frac{\rho_s N(N-1)\Gamma_0^2}{8\pi\Omega} \left(\frac{1-\Omega_0}{\Omega}\right) = 0 . \tag{3.45}$$

So in equilibrium we have $\Omega = \Omega_0$. This means that the vortex array rotates at the same rate as its container.

The next step is to set $\Omega = \Omega_0$ in 3.44 and minimize it with respect to the number of vortices N_v,

$$\frac{\partial F'(\Omega_0)}{\partial N_\mathrm{v}} = \frac{\rho_s \Gamma_0^2}{4\pi} \left(N_\mathrm{v} + \ln\left(\frac{\Lambda}{\xi}\right) - \frac{1}{2} - \frac{2R^2\pi\Omega_0}{\Gamma_0} + 2N_\mathrm{v} \ln\sqrt{\frac{2\pi R^2 \Omega_0}{N_\mathrm{v}\Gamma_0}} \right) = 0 , \tag{3.46}$$

or

$$N_\mathrm{v} = \frac{2R^2\pi\Omega_0}{\Gamma_0} - \ln\left(\frac{\Lambda}{\xi}\right) - \frac{1}{2} + 2N_\mathrm{v} \ln\sqrt{\frac{2\pi R^2 \Omega_0}{N_\mathrm{v}\Gamma_0}} . \tag{3.47}$$

If we neglect all but the first term of this equation we reproduce equation 3.40. This gives the number of vortices that would fill the whole container and is the zeroth order prediction, $N_\mathrm{v}^{(0)} = \frac{2R^2\pi\Omega_0}{\Gamma_0}$. If we take this equation and substitute it back the first order corrections to N_v are found to be

$$N_\mathrm{v}^{(1)} = -\ln\left(\frac{\Lambda}{\xi}\right) - \frac{1}{2} . \tag{3.48}$$

The number of vortices in the container is is corrected by the logarithm of the distance between vortices. This means that as the vortices get packed tighter together, corresponding to and increase in Ω_0, the correction gets smaller.

The energy of the array of vortices is found by subsituting the first order approximation for the number of vortex lines, $N_{\rm v}^{(0)}$, into the free energy 3.44. In equations 3.42 and 3.43 the first order approximation is equivalent to setting $r = R$ where many terms become zero. After this we quickly get the energy of an array of vortices,

$$E' = -\frac{1}{4}\rho_s \pi R^4 \Omega_0^2 + \frac{1}{2}\rho_s \Gamma_0 R^2 \Omega_0 \left(\ln\left(\frac{\Lambda}{\xi}\right) - \frac{1}{2} \right) . \tag{3.49}$$

3.7 Vortex Sheets

Around the same time that Feynman did his work on quantized vortex lines Landau and Lifshitz had another idea [18], based on the work of Onsager and London, for carrying circulation in ^4He. Instead of being confined to single vortices Landau and Lifshitz thought that the circulation might be quantized radially in sheets from the centre of the rotating vessel. Though it turned out to be wrong for ^4He this idea turned out to be right for ^3He and this construction will be used later in chapter 5.

We will once again start with the fact that a superfluid exhibits potential flow. The velocity field for the potential flow is, as in Feynman's case,

$$v_k = \frac{k\hbar}{mr} \text{ where } k = 0, \pm1, \pm2, ... \tag{3.50}$$

One can imagine a number of concentric cylidrical regions of radii r_1, r_2,...,r_n, labeled from the centre out, where in the region between r_k and r_{k-1} the circulation is $\frac{k\hbar}{m}$. As a consequence of this there are discontinuities in the velocity of the fluid at r_k known as vortex sheets. As described earlier a fluid can only have circulation if it is not simply connected. This means that for $r < r_1$ the fluid cannot carry circulation and v_1 is zero.

We will follow Landau and Lifshitz in calculating the energy. In their model they did not restrict the flow between sheets to have quantized circulation but instead assume the flow to have a circulation b_i. Then the superfluid velocity becomes $v_i = \frac{b_i}{r}$. The discontinuities in the velocity carry a surface tension with them of the order $\alpha \sim \frac{\hbar^2}{md^4}$, where d is the atomic spacing, as predicted by Mott [28]. The energy of this system is then the sums of the energy of all the sheets and all the regions of irrotational flow in between them,

$$F = \pi\rho_s \sum_i b_i^2 \ln\frac{r_i}{r_{i+1}} + 2\pi\alpha \sum_i r_{i+1} , \tag{3.51}$$

where the suffix $i = 1, 2, 3, ..$ labels successive layers from the inside outwards. The first term is the kinetic energy of the superfluid and the second term is the energy associated with the surface tension. The fluid in this system also carries angular momentum

$$M = \rho_s \pi \sum_i b_i (r_i^2 - r_{i+1}^2)\,. \tag{3.52}$$

Only the superfluid part of $\rho = \rho_\mathrm{s} + \rho_\mathrm{n}$ has been written here. The normal component of the superfluid acts just like a viscous fluid and has the energy and momentum of a rotating rigid object and is not effected by the presence of layers, so it will be neglected. The free energy of the system given by equation 3.2 is

$$F' = \pi\rho_s \sum_i \left[b_i^2 \ln \frac{r_i}{r_{i+1}} - \Omega b_i (r_i^2 - r_{i+1}^2) \right] + 2\pi\alpha \sum_i r_{i+1}\,. \tag{3.53}$$

We can now minimize this with respect to b_i to find the circulation

$$b_i = \frac{\Omega}{2} \frac{r_i^2 - r_{i+1}^2}{\ln \frac{r_i}{r_{i+1}}}\,. \tag{3.54}$$

Substituting this back into the free energy gives

$$F' = 2\pi\alpha R \sum_i \left[x_{i+1} - \lambda \frac{x_i^2 - x_{i+1}^2}{\ln \frac{x_i}{x_{i+1}}} \right]\,, \tag{3.55}$$

where we have made the dimensionless quantities $x_i = r_i/R$ and $\lambda = \frac{\rho_s \Omega R^3}{8\alpha}$.

We are now free to consider the limiting cases of λ. Since R, ρ_s and α are fixed the case when $\lambda \ll 1$ is associated with slow rotation. It is found that $1 \gg x_1 \gg x_2 \gg ...$ which implies that successive terms of equation 3.53 are much smaller than the preceeding term and that the parameter $\zeta = x_{i+1}/x_i \ll 1$. A single term of the equation 3.53 can be written as

$$F_i' = 2\pi\alpha x_i \left[\frac{\zeta_{i+1} + \lambda x_i}{\ln \zeta_{1+i}} \right]\,. \tag{3.56}$$

Minimizing with respect to ζ_{i+1} gives the condition

$$\sqrt{\zeta_{i+1}} \ln \frac{1}{\sqrt{\zeta_{i+1}}} = \frac{\sqrt{\lambda x_i^3}}{2}\,. \tag{3.57}$$

For non-zero rotation there exist sheets which are concentrated at the centre of the container. The maximum radius for a sheet is given when $\zeta = x_1$.

The opposite case is fast rotation where $\lambda \gg 1$. Here the layers are evenly distributed in the fluid. If the space between layers is h, in the limit of $h \to 0$, the fluid should mimic that of a solid body with energy

$$F' = -2\pi\alpha R\lambda \sum_i (x_i^4 - x_{i+1}^4)\,. \tag{3.58}$$

Subtracting this from equation 3.53, series expanding in $\Delta = h/R$, and multiplying the energy by the number of sheets $1/\Delta$ we get the energy for all the sheets,

$$F_{\text{sheets}} = 2\pi\alpha R \left[\frac{x}{\Delta} + \frac{4\lambda}{3} x \Delta^2 \right] . \tag{3.59}$$

Minimizing this with respect to Δ and restoring the dimensionless parameters gives the spacing between the sheets,

$$h = \left(\frac{3\alpha}{\rho_s \Omega^2} \right) . \tag{3.60}$$

It is this equation which has been verified for ^3He and is the triumph of this model [42].

Chapter 4

Neutron Stars

The average neutron star has a mass of 1.4 $M_\odot$ contained in a radius of only 10 km giving it the immense density of 10^{15} g/cm^3 [12]. The study of neutron stars has its beginning in the proposal of a nucleus star by Landau in 1932. Later, and more concretely, Baade and Zwicky suggested that neutron stars are formed in supernovae in which the iron core of a massive star exceeds the Chandrasekhar limit and collapses. The subsequent identification of radio pulsars as neutron stars is what sparked real interest in understanding these objects further. Since then, nearly 1200 objects have been identified as pulsars.

The name pulsar comes from the periodic pulses of radiation these stars emit. Their magnetic and rotational axes are misaligned causing them to emit dipole radiation which appears to pulse on and off as the star rotates similar to a light house beacon. The energy lost to this radiation is

$$\dot{E} = I\Omega\dot{\Omega} = \frac{B^2 R^6 \Omega^4 \sin(\theta)^2}{6c^3}\,, \tag{4.1}$$

which causes the star to gradually rotate slower and slower [12]. The typical moment of inertia is $I \sim 10^{45}$ g cm^2, the magnetic field is of the order $B \sim 10^{12}$ G and the period of rotation $P = \frac{2\pi}{\Omega}$ ranges from 1.5 ms to 8.5 s.

Neutron stars have been observed to have rapid increases in their period of rotation of the order $\Delta P/P \sim 10^{-6}$. These rapid changes are called glitches and a more detailed discussion of them can be found further in this chapter. Glitches are the motivation behind the standard model in which a neutron star has a solid crust and a liquid interior. This model has been simplified and presented in figure 4.1.

To investigate the internal structure further we are required to look at the nature of nuclear matter at high densities and low temperatures. Neutron stars have the distinction of being extremely cold in statistical mechcanical terms which points to the possibility that the nuclear matter in the core is a condensate. The challenge in studying neutron stars is trying to come up with a picture which describes all these phenomena. Much of this chapter will focus on developing the standard picture of the interior of a neutron star. A lovely review is provided by J.A. Sauls [33] and the most relevant points will be outlined in the first part of this chapter.

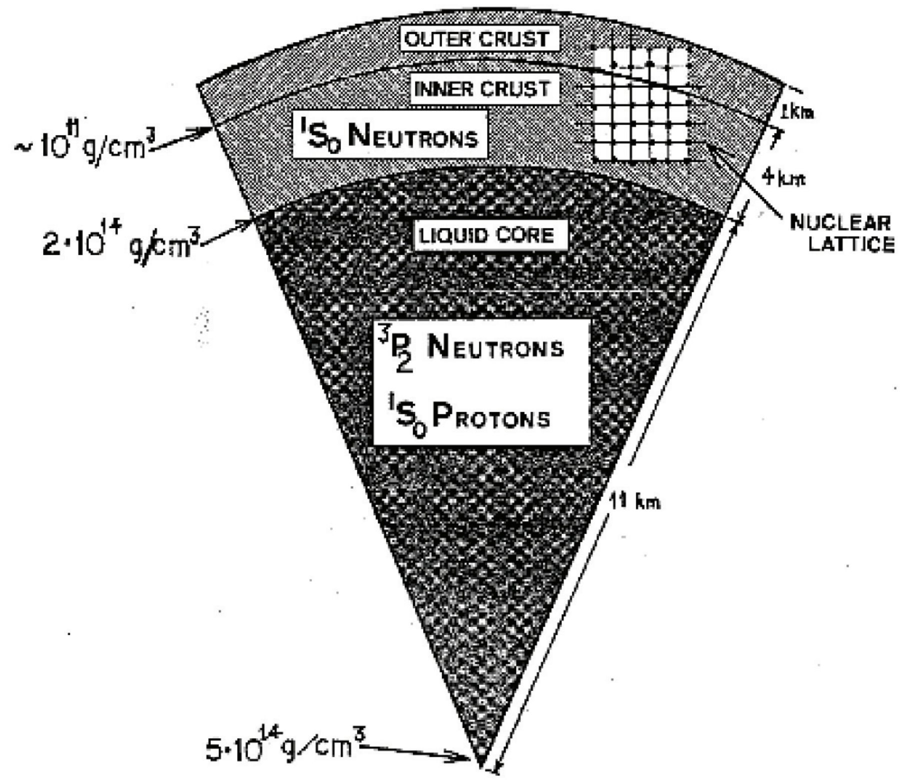

Figure 4.1: Layered structure of a neutron star.

4.1 Structure of a Neutron Star

In reviewing the structure of a neutron star it is necessary to look at the states of nuclear matter and how they behave at high densities. Being fermions, and subject to the Pauli exclusion principle, neutrons and protons can only condense by forming pairs known as Cooper pairs. The wave function of a Cooper pair is described as $\psi_{s_1,s_2}(\mathbf{R},\mathbf{r})$, where s_1, s_2 are the spin projections of each particle and R and r are the centre of mass coordinate and orbital coordinates respectively.

Cooper pairs orbit on the order of $r \sim 100$ fm, which seems small, but is large compared to the average distance of neutrons in the star. This means that a Cooper pair is not actually a pair of particles in the traditional sense, but are a pair in momentum space. This pairing of fermions now makes a boson which is no longer subject to the Pauli exclusion principle. It is now possible for a number of Cooper pairs to occupy the same energy state. This ordering is called a condensate. For a Cooper pair this means that the amplitude of its wave function is coherent over macroscopic distances and a large number of neutron pairs are described by identical wave functions.

While there is no direct evidence that the core of a neutron star contains superfluid neutrons and superconducting protons, a good argument can be made for it based on experiments conducted on Earth and predictions from the BCS theory of superconductors. The argument uses the fact that neutron stars are very cold, having a temperature $T \sim 10^8 K \ll T_{\text{fermi}} \sim 10^{12}\text{K}$, and that the transition temperature for nuclear matter to condense here on earth is $T_{\text{c}} \sim 10^{-3} T_{\text{fermi}}$ [33].

For fermions to form Cooper pairs, and in turn make a condensate, it is necessary for there to exist an attractive force between them, nop matter how small [2]. Much work has been done studying the nucleon-nucleon interactions which form Cooper pairs.

Figure 4.2[1] shows the phase shifts of free neutron scattering complied by Tamagaki. A positive phase shift implies an attractive interaction between nucleons. Figure 4.2[1] shows that the most dominate states are ${}^1S_0$ neutrons at lower densities and ${}^1S_0$ protons and ${}^3P_2$ neutrons at higher densities [33] [35].

It is this dependence on density which gives the layered structure of the neutron star. As the density increases towards the centre of the star the state most likely to pair to form a condensate changes. Near the crust the interaction between neutrons in the ${}^1S_0$ state is the strongest and forms the condensate while towards the core the ${}^1S_0$ for protons and the ${}^3P_2$ state for neutrons have stronger interactions and condense.

Though the strength of the interaction is not important in the formation of a Cooper pair, it does give an indicator of the critical temperature of the nuclear matter. Data for the transition temperatures for each phase is shown in figure 4.2[1]. This clearly shows transition temperatures $T_c \gtrsim 0.5$ MeV $\approx 5^{10}$ K which are all greater than the temperature of a neutron star. It is quite likely then that the nuclear matter in a star is a condensate.

4.1.1 States of Nuclear Matter

In earlier chapters we described vortices using a single complex scalar order parameter, but because of spin projections, Cooper pairs can have quite complicated order parameters. We will review the states which are found in a neutron star, figure 4.1[1], and see in what cases these can be reduced to single order parameters. This reduction will allow us to use the theory from earlier chapters to describe what is happening inside a neutron star.

The ${}^1S_0$ state is described by an order parameter with total spin $|S| = |s_1 + s_2| = 0$, a spin singlet, and orbital angular momentum $L = 0$. The spins are paired such that the state is magnetically neutral and the angular momentum has a spherical symmetry. Because of this the amplitude of the ${}^1S_0$ state can be reduced to a single complex scalar field, $\psi(\mathbf{R}) = \psi_{\downarrow,\uparrow}$. The

[1] Figures 4.2 and 4.1 from [33] were obtained directly from J.A. Sauls.

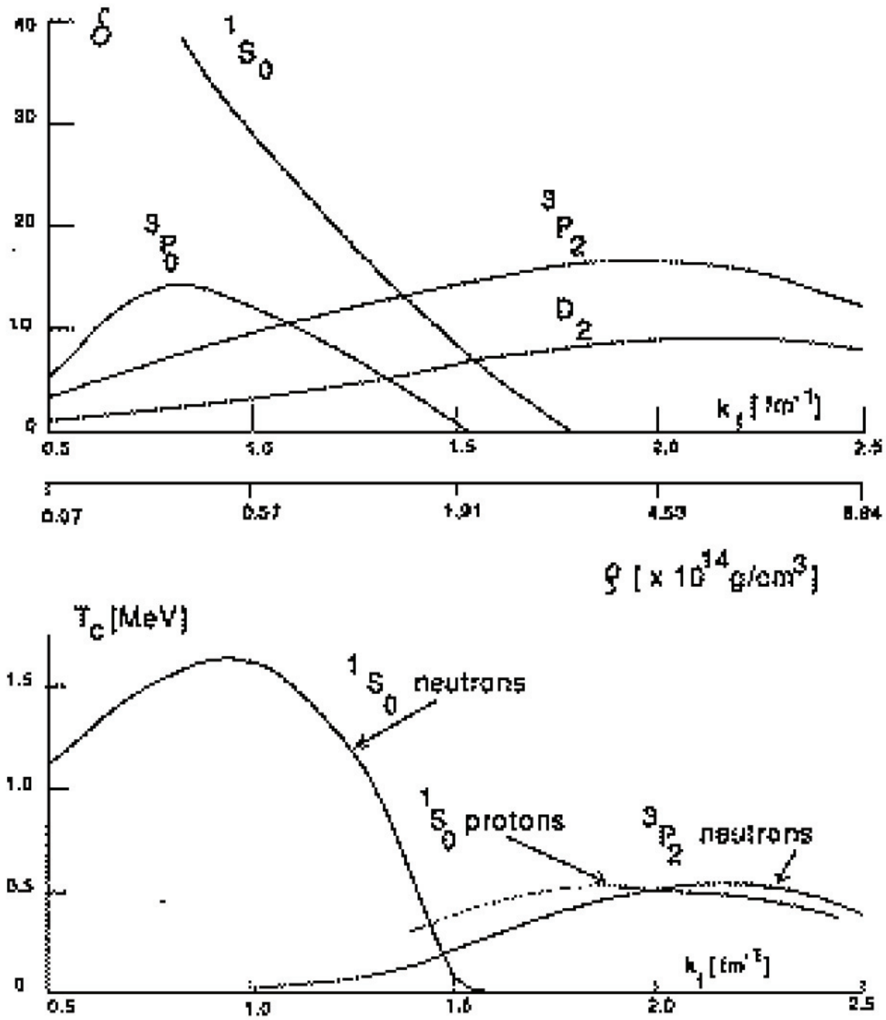

Figure 4.2: Phase shift and critical temperature for different states of nuclear matter.

behaviour of the superfluid neutrons in the crust and superconducting protons in the core can be described using the theory that was developed for ^4He and explained in previous chapters.

The dominant state at high densities is the $^3P_2$ state. The wave function is a spin triplet state with total angular momentum $J = 2$ and can be written as

$$|\psi\rangle = \sum_{J_z=0,\pm1,\pm2} \psi_{J_z} |J = 2, J_z\rangle \,. \tag{4.2}$$

When the state is not rotating $J_z = 0$ represents the ground state. This means that far away from vortices we can describe the state of the superfluid with a single order parameter ψ_0. Close to the core of the vortex it is necessary to use all spin states to describe the superfluid. Qualitatively this does not change the velocity distribution or the behaviour of the order parameter moving away from the vortex but it does magnetize the core of the vortex.

4.2 Vortices in Neutron Stars

Having described the nature of matter in a neutron star we can now apply the results from earlier chapters specifically to neutron stars.

Following the argument in chapter three, the rotation of the neutron star will cause a triangular array of vortices in the neutron superfluid. These form parallel to the axes of rotation and have a areal number density given by formula 3.40,

$$n_v = \frac{2\Omega}{\Gamma_0} = \frac{\Omega m}{h} \sim 10^{10}\ m^{-2} \,. \tag{4.3}$$

From this the average vortex line spacing is,

$$l_v = n_v^{-\frac{1}{2}} \sim 10^{-5}\ m \,. \tag{4.4}$$

If there is a magnetic field that penetrates the proton superconductor it does so by forming flux tubes with a density given by

$$n_f = \frac{B}{\Phi_0} = \frac{Bq}{hc} \sim 10^{16}\ m^{-2} \,, \tag{4.5}$$

which means the spacing between the proton vortices is,

$$l_f = n_f^{-\frac{1}{2}} \sim 10^{-8} m \,. \tag{4.6}$$

Notice that there is a much higher density of proton vortices than neutron vortices. It is now easy to see how the neutron vortices could become entangled in the proton vortices, meaning they could only move as a single object.

The type of superconductor the protons are expected to make is given by the Landau parameter. The flux tubes are made of normal protons and the radius of the flux tubes is given by the proton coherence length $\xi_p \sim 30$ fm and the London penetration depth for a proton superconductor is $\lambda_p \sim 80$ fm [22]. Using these values we can calculate the Landau parameter for nuclear matter in a neutron star,

$$\kappa = \frac{\lambda_p}{\xi_p} \sim 2 \,. \tag{4.7}$$

Comparing with 2.52 we see that this cursory glance indicates that the protons form a type II superconductor and the flux tubes do indeed form a lattice. It is expected that the flux tubes have a complicated twisted structure and that the superfluid neutron vortices have many proton vortices tangled around them.

4.3 Glitches

Observations of the rotation of neutron stars show them slowly and constantly decelerating due to the energy lost from the dipole radiation. In 1969 the Vela pulsar was seen to suddenly speed up rotation with a change in angular velocity $\Delta\Omega/\Omega \sim 2 \times 10^{-6}$. Following this there was a discontinuous increase in the deceleration until the rate of deceleration returns to its pre-glitch value. In the Vela pulsar this relaxation time is of the order $\tau \sim 1$ years. Similar phenomena has been seen with the Crab pulsar, though the glitches are much smaller $\Delta\Omega/\Omega \sim 10^{-8}$ and the relaxation times are much shorter on the order $\tau \sim 4$ days [29].

What causes glitches is not well understood. Two models will be introduced in this section, one which describes the small glitches found in the Crab nebula and one which could describe the larger ones in the Vela nebula.

The small glitches in the Crab nebula could be caused by a phenomena known as starquakes [31]. Because the neutron star is rotating quickly centripetal forces could tend to make the star oblate when it forms. As the star slows down it wants to become more spherical, but the crystalline crust formed on the oblate surface will not allow this to happen continuously. Instead the star will keep its shape until the strain becomes too great and the crystalline crust fractures, making the star more spherical. This moves the mass closer to the axis of rotation and, because of conservation of angular momentum, the star spins faster. Detailed analysis of this model has been done and it can only support the relaxation times seen in the Crab nebula, not in the Vela, so another mechanism has to be introduced.

The second mechanism is one based on the metastable flows observed in liquid helium [29]. Because of the Feynman-Onsager relation a container that is decelerating must lose neutron vortices at a rate

$$\dot{N}_\mathrm{v} = 4\pi R^2 \frac{m_\mathrm{n}}{h} \dot{\Omega} \,. \tag{4.8}$$

For a neutron star this is of the order $N_\mathrm{v} \sim 10^{10}$ per day. If there are impurities on the wall of the container, instead of being annihilated on the wall, the vortices will become pinned to the vessel. The presence of these vortices means that the superfluid cannot slow down with the container and that a metastable flow is established. When a vortex de-pins the superfluid will suddenly slow down and its angular momentum will be transferred to the crust. Because the angular momentum must be conserved for a free body a sudden decrease in the angular velocity of the core will be accompanied by an increase in the angular velocity of the crust.

There is a problem with this mechanism though [33]. To change the rotation of such a large, dense object requires energies of the order $\Delta E_{\rm rot} = 2\frac{\Delta\Omega}{\Omega} \sim 10^{43}$ erg. Observations show that the change in angular acceleration resulting from the glitch implies that the change in the moment of inertia of the stars crust is $\Delta I/I = \Delta\Omega/\Omega \sim 10^{-2}$, which means that about 10^{13} neutron vortices would have to simultaneously de-pin. There is no mechanism to explain this mass unpinning.

Part II

Consequences of Currents in Vortices

Chapter 5

Current in Vortices

5.1 Motivation

The second half of this thesis is motivated by the recent discovery that neutron stars precess [13] and that calculations by Link [22] show that the magnitude ($\sim 3^\circ$) and frequency (~ 1 per year) of this precession conflicts with the current picture of vortices in a neutron star.

In the previous chapter it was established that neutron stars form both neutron and proton vortices. In the standard picture both condensates are subjected to angular momentum and magnetic flux and form lattices, but the proton vortices are much more numerous than the neutron vortices and are tangled around them. When the proton condensate rotates it does not form a vortex lattice but corotates with the crust at the expense of a London current. The existence of precession means that the neutron vortices no longer form along the rotational axis of the star, but along the axis which is the sum of the precession and angular momentum vectors. When the star precesses the vortices now move with respect to the rotation of the star and, in turn, with respect to the proton vortices which are entangling them. If the precession is large enough one of two things must happen; either the flux tubes move with the neutron vortices or the neutron vortex and the proton vortex move through each other.

If the neutron and proton vortices are required to move together then there are severe restrictions on the precession. Because the core of the star is superconducting the proton vortices, which carry magnetic flux, are resistant to being moved and thus the neutron vortices are restricted to move slowly. This means that the neutron vortices are pinned to the rotation of the protons and to the crust. Link has clearly found that if this pinning is present the neutron star can only precess at high frequencies $\omega \sim 10$ rad/s. For the star to precess more slowly at large amplitudes it is necessary for the neutron vortices to pass through the flux tubes.

If the precession has large amplitudes, such as those observed, it is possible for the star to have the energy to pass large numbers of neutron vortices through proton vortices. This is a highly dissipative process. When a neutron vortex passes through a flux tube excitations known as kevlons propagate along the vortex. The energy lost due to the creation of a single kevlon is very small, but when the vast number of vortices are taken into account it grows quickly. Link calculates the dissipation rate to be $\dot{E} \sim 10^{41}$ ergs/s. In comparison the rotational energy of a neutron star is only $E_{\rm rot} \sim 10^{44}$ erg. This gives a dampening time of the precession to be smaller than 1 hour. When the precession is small the vortices can no longer pass through each other and the precession is limited again to $\omega \sim 10$ rad/s.

This leads to the conclusion that either the protons and neutrons do not coexist as a condensate or that the proton condensate does not form a lattice but but some other structure. It is very unlikely that protons and neutrons don't co-exist in the star as condensates so this paper will focus on the structure of the vortices. If the proton vortices are to not form a lattice they must have a mechanism to make the interaction between them attractive. As shown in chapter 4 the nature of the nuclear matter in the star explicitly points to type II superconductivity which implies the vortices should repel. It is necessary then to look for a new term in the interaction force to make vortices attract.

5.1.1 Previous Work

Sedrakian [34] has shown that the existence of a type I superconductor in a neutron star indeed resolves the conflict in Link's paper. He assumes that the equilibrium structure for a type I superconductor is a set of superconducting and normal domains. By use of a hydrodynamic restriction based on the moment of inertia of the crust and the moment of inertia of the superfluid he showed that the alternating domain structure seen in type I superconductors will always allow undamped precession. We can look at this another way as well. Link's argument relied on the proton vortices tangling around the neutron vortices. In a domain structure there is more room for the neutron vortices to move unhindered by the proton vortices, and large amplitude, high frequency precession would be allowed.

Microscopic mechanisms for type I superconductivity have been studied before. Buckley et al [15] have shown that the asymmetry in the scattering length of the neutron and proton cooper pairs can add a factor to the coherence length of the proton vortices that makes the actual value for the coherence length much larger. This increased coherence length changes the Landau-Ginzburg parameter meaning that the regimes for type I and type II superconductivity can be quite different. Using typical values found in a neutron star this new Landau-Ginzburg parameter indicates that there should be type I superconductivity in the star.

In what follows we will suggest a new mechanism which leads to a superconductor which behaves like a type I even when the parameters of the system suggest that it is type II.

5.2 Current and Vortices

Suppose there are two wires, placed parallel to each other, carrying current. We know from electromagnetism that if the current runs in the same direction, the wires will be attract to one another. Now consider that superconducting vortices, instead of wires, are carrying the current. There are three forces working against each other: the attractive electromagnetic force from the current, the repulsive electromagnetic force from the gauge field, and the attractive force from the order field. If the current were strong enough, the attractive force it generates would make it so the vortices always attract and the superconductor would act like it was type I.

Currents in vortices could also provide a source for a toroidal magnetic field in the neutron star resulting in magnetic helicity. It has been argued that a toroidal component in the magnetic field of a neutron star is necessary for stablility of the poloidal magnetic field [23] and that it could describe the temperature distribution of the crust [40]. In our model, if current traveled along the core of the vortex, exited one end, traveled along the crust of the neutron star and entered at the other end of the vortex, a large current loop would be created. A number of vortices would create a number of current loops, and a toroidal magnetic field would appear. This toroidal field, combined with the poloidal field present in a neutron star, would create a non-zero magnetic helicity.

Though the idea of currents in vortices was introduced by Witten [43] in the context of cosmic strings, we are interested in the recent developments in QCD [36] [25] [27] and condensed matter [3] [41]. For our applications in neutron stars we are specifically interested in the current derived by Metlitski et al [27],

$$j = \frac{e\mu}{2\pi^2}\Phi\,. \tag{5.1}$$

where μ is the chemical potential and Φ is the magnetic flux.

Concerning ourselves with the derivation and existence of these currents is beyond the scope of this work but some notes on the nature of the current are are required. In their paper Metlitski et al. derive 5.1 as an axial current using a chiral Lagrangian. In the case of an axial current, $\mu = \mu_L + \mu_R$, where μ_L and μ_R are the chemical potentials corresponding to two reservoirs of particles with different chirality. For our purposes an axial current will not work as it is not capable of being a source. To have a current source we want to look at a vector current where $\mu = \mu_L - \mu_R$. Usually, and in the case of Metlitski et al., $\mu_L = \mu_R$, which leads to an vector current of zero. To get a non-zero current it is necessary to break P-parity so that $\mu_L \neq \mu_R$. This is not an unreasonable requirement in a neutron star as P-parity is broken in π and kaon condensation, which are some of the proposed states of nuclear matter at high densities.

Looking at equation 5.1 we also see that for there to be current there must be magnetic flux present. In our case this is particularly the quanta of flux which is carried in a superfluid vortex with winding number $n = 1$,

$$\Phi_0 = \frac{2\pi\hbar c}{q}\,. \tag{5.2}$$

The fact that the flux is confined to the core of the vortex also means that the current is confined to the core of the vortex.

It is worth noting that Alexseev et al. [3] showed that a similar current can arise from condensed matter arguments. The basic requirement is to have two commuting, conserved charges, Q_L and Q_R which are associated with two external reservoirs with chemical potentials where $\mu_L \neq \mu_R$. They derive a universal current formula and when it is applied to a massless Dirac field, a current very similar to 5.1 is derived where zero modes in the system to carry the current when the system is placed in a background electromagnetic field.

We will now derive the interaction between two superconducting vortices[1] where a normal electromagnetic current is present in their cores. A substitution of 5.1 into this interaction will determine at what chemical potentials superconducting vortices will always act like type I, rather than being subject to parameters which determine type I and type II regimes.

5.2.1 Interaction between superconducting vortices carrying current

To calculate the interactions we will use the same technique used in chapters 2 and 3 in which we reduce the theory to a non-interacting, linear one and model the vortices as point sources. The interaction energy is then easily calculated from this linear theory. We start with the Landau-Ginzburg free energy with a current source $\mathbf{j} = j\delta^2(\boldsymbol{r})\hat{\boldsymbol{z}}$ added,

$$\begin{aligned} E &= \int d^2x \left\{ \frac{\hbar^2}{2m} \left| \left(\nabla - \frac{iq\mathbf{A}(\mathbf{x})}{\hbar c} \right) \psi(\mathbf{x}) \right|^2 - \mu \left| \psi(\mathbf{x}) \right|^2 \right. \\ &\left. + \frac{a}{2} \left| \psi(\mathbf{x}) \right|^4 + \frac{1}{8\pi} \left(\nabla \times \mathbf{A}(\mathbf{x}) \right)^2 + \mathbf{j} \cdot \mathbf{A} \right\} . \end{aligned} \tag{5.3}$$

This energy can be minimized to yield the the equations of motion. Minimizing with respect to the vector potential $\mathbf{A}$ gives us

$$\frac{1}{4\pi} \left(\nabla^2 \mathbf{A} - \nabla (\nabla \cdot \mathbf{A}) \right) = -\frac{\hbar q}{m} \mathbf{j}_{\text{Noether}} - \mathbf{j} \,. \tag{5.4}$$

The right hand side is written in terms of a Noether current [30]

$$\mathbf{j}_{\text{Noether}} = \frac{1}{2i} \left(\psi^\dagger \nabla \psi - \psi \nabla \psi^\dagger \right) - \frac{q}{\hbar c} \mathbf{A} \left| \psi \right|^2 . \tag{5.5}$$

Minimizing with respect to the order field gives

$$\frac{\hbar^2}{2m} \left(\nabla - \frac{iq}{\hbar c} \mathbf{A} \right)^2 \psi = a \left| \psi \right|^2 \psi - \mu \psi \,. \tag{5.6}$$

We are looking for defects so we will no longer assume that $\psi(x)$ is constant. To decouple our set of differential equations it is convenient to define

$$\psi = \sqrt{\frac{\mu}{a}} \rho(r) e^{i\phi} \,, \tag{5.7}$$

[1] A nearly indentical calculation can be carried out for the interaction between superfluid vortices carrying current. The results are outlined in Appendix C

$$\mathbf{A} = \frac{\hbar q}{c}\frac{a\left(r\right)}{r}\ \hat{\boldsymbol{\phi}} + f(r)\ \hat{\boldsymbol{z}}\,, \tag{5.8}$$

where $\rho\left(r\right),\ a\left(r\right) \to 1$ as $r \to \infty$ and $\rho\left(r\right),\ a\left(r\right) \to 0$ as $r \to 0$ and r and ϕ the cylindrical coordinates. The phase of $\psi(x)$ is chosen to mimic a vortex with winding number $n = 1$. We can further define

$$\rho\left(r\right) = 1 + \sigma\left(r\right)\ , \tag{5.9}$$

$$a\left(r\right) = 1 + r\alpha\left(r\right)\ , \tag{5.10}$$

such that $\sigma\left(r\right),\ \alpha\left(r\right) \to 0$ as $r \to \infty$. Substituting 5.7 and 5.8 into equation 5.4 and linearizing yields the two equations,

$$\frac{\partial^2\alpha}{\partial r^2} + \frac{1}{r}\frac{\partial\alpha}{\partial r} - \left(\frac{1}{r^2} + \frac{1}{\lambda^2}\right)\alpha = 0\,, \tag{5.11}$$

$$\left(\nabla^2 - \frac{1}{\lambda}\right)f(r) = 4\pi j\delta^2(\mathbf{r})\,, \tag{5.12}$$

where λ is the London penetration depth we derived earlier. The first equation is the modified Bessel equation of the first order. We want a solution that goes to zero as $r \to \infty$ so we choose the solution to be a modified Bessel function of the second kind, $\alpha = \frac{1}{\lambda}K_1\left(\frac{r}{\lambda}\right)$. The second equation is just a statement of the Green's function,

$$(\nabla^2 - \alpha^2)K_0(\alpha r) = -2\pi\delta^2(\mathbf{r})\,. \tag{5.13}$$

Going back through all the substitutions and then using a gauge transformation we find that the vector potential is

$$\mathbf{A} = \frac{\hbar c}{q\lambda}K_1\left(\frac{r}{\lambda}\right)\ \hat{\boldsymbol{\phi}} - 2jK_0\left(\frac{r}{\lambda}\right)\ \hat{\boldsymbol{z}}\,. \tag{5.14}$$

A similar procedure follows for the solution to the order parameter. Substituting 5.7 and 5.10 into 5.6 and linearizing yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\sigma}{\partial r}\right) = \frac{4m\mu}{\hbar^2}\sigma\,. \tag{5.15}$$

This is a modified Bessel equation of the zeroth order. The solution is

$$\sigma(x) = K_0\left(\frac{\sqrt{2}}{\xi}r\right)\ , \tag{5.16}$$

where $\xi = \sqrt{\frac{\hbar^2}{2m\mu}}$ is the coherence length.

Now that we have solutions to the equations of motion we can calculate the interaction energy. The idea is to reduce the theory to a non-interacting, linear one and model the vortices as point sources. The interaction energy is then easily calculated from this linear theory. To make it make this easier it is useful to remove the phase in ϕ by writing it as $\sqrt{\frac{\mu}{a}}(1-\sigma)$. To linearize the theory we expand in ρ and $\mathbf{A}$ and keep only quadratic terms to get

$$E_{\text{free}} = \int d^2x \left\{ \frac{\mu}{a}\frac{\hbar^2}{2m}(\nabla\sigma)^2 + \frac{1}{8\pi}\left((\nabla\times\mathbf{A})^2 + \frac{\mathbf{A}^2}{\lambda^2}\right) + 2\frac{\mu^2}{a}\sigma \right\} . \quad (5.17)$$

The source terms are

$$E_{\text{source}} = \int d^2x \left\{\tau\sigma + \boldsymbol{\mathcal{J}}\cdot\mathbf{A}\right\} , \quad (5.18)$$

where τ and $\mathcal{J}$ are the sources for the fields σ and $\mathbf{A}$. Minimizing this we get the equations of motion,

$$\left(\nabla^2 - \frac{2}{\xi^2}\right)\sigma = \frac{a}{\hbar^2}\frac{a}{\mu}\tau , \quad (5.19)$$

$$\left(\nabla^2 - \frac{1}{\lambda^2}\right)\mathbf{A} = 4\pi\boldsymbol{\mathcal{J}} . \quad (5.20)$$

We want to solve for the sources $\boldsymbol{\mathcal{J}}$ and τ such that σ and $\mathbf{A}$ have the same asymptotic solutions we obtained earlier in 5.14 and 5.16. Using 5.13 and the derivative of 5.13 we can solve for the sources,

$$\tau = -\frac{\hbar^2}{m}\frac{\mu}{a}2\pi\delta^2(\mathbf{x}) , \quad (5.21)$$

$$\boldsymbol{\mathcal{J}} = \frac{\hbar c}{2q}\frac{\partial\delta^2(\mathbf{x})}{\partial x}\ \hat{\boldsymbol{\phi}} + j\delta^2(\mathbf{x})\ \hat{\boldsymbol{z}} . \quad (5.22)$$

The interaction energy is found by substituting $\boldsymbol{\mathcal{J}} = \boldsymbol{\mathcal{J}}_1 + \boldsymbol{\mathcal{J}}_2$, $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$, $\tau = \tau_1 + \tau_2$ and $\sigma = \sigma_1 + \sigma_2$ into the total energy $E = E_{\text{free}} + E_{\text{source}}$ and subtracting of the energies of the vortices, leaving only cross terms. The subscripts 1 and 2 refer to two separate vortices and positions $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. Using the equations of motion we get left over cross terms left which are interpreted as the interaction energy;

$$E_{\text{int}} = \int d^2x \left\{\tau_1\sigma_2 + \boldsymbol{\mathcal{J}}_1\cdot\mathbf{A}_2\right\} . \quad (5.23)$$

Though it doesn't look it, the interaction energy is symmetric in the exchange of the subscripts 1 and 2. The apparent asymmetry arises when the equations of motion for either subscript 1 or 2 are substituted in. Using 5.8, 5.9, 5.21 and

5.22 the interaction energy can be written,

$$\begin{aligned} E_{\rm int} &= \int d^2x \left\{ -\frac{\hbar^2}{m}\frac{\mu}{a} 2\pi\delta^2(\mathbf{x}-\mathbf{x}_1) K_0\left(\frac{\sqrt{2}}{\xi}(\mathbf{x}-\mathbf{x}_2)\right) \right. \\ & -\frac{\hbar^2 c^2}{2q^2\lambda}\frac{\partial\delta^2(\mathbf{x}-\mathbf{x}_1)}{\partial x} K_1\left(\frac{\mathbf{x}-\mathbf{x}_2}{\lambda}\right) \\ & \left. -2j_1 j_2 \delta^2(\mathbf{x}-\mathbf{x}_1) K_0\left(\frac{\mathbf{x}-\mathbf{x}_2}{\lambda}\right) \right\} , \\ &= \frac{1}{2}\left(\frac{\hbar c}{q\lambda}\right)^2 \left[\left(1-\frac{4q^2\lambda^2 j_1 j_2}{\hbar^2 c^2}\right) K_0\left(\frac{d}{\lambda}\right) - K_0\left(\frac{\sqrt{2}d}{\xi}\right)\right] , \end{aligned} \tag{5.24}$$

where $d = |\mathbf{x}_1 - \mathbf{x}_2|$. Comparing this with 2.50 we see that the first two terms are identical to the interaction between gauge vortices without current. The only new piece is the third term which comes directly from the current. If j_1 and j_2 run in the same direction there is and attractive force and if they run in opposite directions there is a repulsive force. This is the expected result if we considered parallel wires carrying current.

The interaction energy 5.24 determines whether the vortices attract or repel and whether we see type I or type II behavior in the superconductor. If we set $j_1 = j_2 = j$ then there are two cases to explore; one when $j < \frac{\hbar c}{2q\lambda}$ and one when $j > \frac{\hbar c}{2q\lambda}$. In the first case, when $j < \frac{\hbar c}{2q\lambda}$, the first term of 5.24 is positive and we obtain the canonical behavior for a superconductor where the Landau-Ginzburg parameter (see 2.52) decides whether the system exhibits type I or type II behavior.

For a solution to the problem raised by Link, the interesting case occurs when

$$j > \frac{\hbar c}{2q\lambda} , \tag{5.25}$$

and the first term in 5.24 becomes negative. This means that there is no longer a tug of war between the gauge field and the order field to determine the type of superconductivity; the vortices always attract. Then there could be systems with a Landau-Ginzburg parameter which indicates the vortices should repel and the superconductor is type II but there is sufficient current for the vortices to attract and act like a type I superconductor.

5.3 Discussion

With currents in vortices we have found a mechanism which reconciles the condractiction between the precession of neutron stars and the belief that there is type II superconductivity inside a neutron star. In calculating the Landau-Ginzburg parameter for a neutron star it is clear (see chapter 4) that the proton superconductor should be type II. Link showed that recent precession observations mean that the superconductor cannot be type II. A sufficient current running along the core of the vortex allows the vortices to attract even if the Landau-Ginzburg parameter indicates they should repel, resolving this problem.

It is logical to ask how big does a current have to be such that it is considered sufficient. The currents in vortices haven't been measured yet so it is actually more constructive to rephrase this in terms of parameters we are familiar with. Consider a superfluid vortex which carries a a single quanta of flux. The current 5.1, in units $\hbar = c = 1$, becomes

$$j = \frac{\mu}{\pi} \,. \tag{5.26}$$

Substituting this into 5.25 gives the inequality,

$$\mu > \frac{\pi}{\lambda q} \,, \tag{5.27}$$

where μ is the chemical potential of the condensate, q is the charge and λ is the Meissner penetration depth. We can see that the inequality is more likely to be satisfied by dense materials with higher chemical potentials such as those found in neutron stars. This bodes well for the current as a mechanism for creating type I superconductors in neutron stars.

Chapter 6

Future Work

In this thesis we have investigated the effect of currents in vortices and found a mechanism which resolves the contradiction raised by Link [22] that the observed precession of a neutron star [13] is not allowed if the superconductor in a neutron star is type II. The mechanism is to allow a current to travel along the superconducting vortices. If this current is large enough the vortices will always attract, regardless of whether the Landau-Ginzburg parameter indicates type I or type II supercondtivity. It is believed that this effect becomes more likely as the density of the condensate increases.

This has great consequences inside a neutron star. Where before superconducting vortices would form a lattice which would get tangled with the superfluid vortices, they could now bunch together and leave space for the superfluid vortices to move unhindered.

Sedrakian [34] showed that the presence of a type I superconductor would allow a neutron star to precess at frequencies and magnitudes that are not allowed when the neutron vortices are entangled in proton vortices. But this freedom could also allow the superfluid vortices to do more novel things. Mastrano et al [24] have presented calculations in which vortices near the crust are grabbed and bundled together by the Kelvin-Helmoltz waves created by the instability which arises when there is a shear stress between two fluids. This bundling of vortices could twist them so they no longer line up in an array, but instead form vortex loops, or vortons.

Vortons made from superfluid vortices are not typically stable [26] but in this chapter we will show how the presence of a current in the core of a vortex and an external magnetic field could make vortons stable. We go further and show that if there are stable vortices then they will attract to each other to form a column of vortons. Because the vortons attract very close to each other this structure would look cease to look like a bunch of individual vortices but like a single cylindrical vortex sheet which carries a surface current similar to a solenoid.

A vorton formed from a superconducting vortex would carry magnetic flux inside it and could add a toroidal component to magnetic field of a neutron star which is apparently required by observations [40]. A vorton carrying current would be most stable if its dipole moment was parallel to the magnetic field of the neutron star. The direction of the flux in the vorton would contribute to a toroidal flux inside the neutron star.

This chapter only goes so far as to propose the idea of vortex sheets and show that such an idea is plausible. The treatment of vortex sheets with currents and the comparison against the canonical vortex array solution will be left for a later paper.

6.1 Vortex Loops with currents in the core

Normally vortons are very unstable becasue of the high energy per unit length of a vortex means the loop wants to collapse. The addition of a current would allow for a vorton to be stable if it is placed in an external magnetic field, $B_{\text{ext}}\ \hat{\boldsymbol{z}}$, and we account for the dipole interaction. In this case the free energy of the system is

$$E'_{\text{vorton}} = E_{\text{vorton}} - \mathbf{\Omega}\cdot\mathbf{M} - \mathbf{m}\cdot\mathbf{B}_{\text{ext}}, \tag{6.1}$$

where $\mathbf{M}$ is the systems angular momentum, $\mathbf{\Omega}$ is the systems angular velocity, and $\mathbf{m}$ is the magnetic moment of the current loop.

For ease we will discuss superfluid vortices which carry current which forms a loop in the xy-plane. In forming a loop there are two components to the energy, that coming from the superfluid E_{sf} and that coming from the current E_{current}. The energy of the current has two components as well, one from the magnetic field and one mechanical energy carried by the particles that make up the current, $E_{\text{current}} = E_{\text{mech}} + E_{\text{EM}}$. The mechanical portion can be found by recognizing that for a loop the current in a non-relativistic regime is $j = \frac{Nqv}{2\pi r_i}$, where N is the number of particles, q is their charge and v is their velocity. Then

$$E_{\text{mech}} = \frac{1}{2}Nm_q v^2 = 2\pi^2\frac{m_e j^2 r_i^2}{dNq^2}. \tag{6.2}$$

The magnetic energy is just

$$E_{\text{EM}} = 2\pi r\frac{\mu_0 j^2}{16\pi}. \tag{6.3}$$

Add all these energies together and we get the energy of the vorton.

$$E_{\text{vorton}} = 2\pi^2\frac{m_e j^2 r^2}{Nq^2} + 2\pi r\frac{\mu_0 j^2}{16\pi} + 2\pi^2 r\rho_s\Gamma_0^2\ln\frac{\Lambda}{\xi} \tag{6.4}$$

We will assume that the vorton is charge neutral so there is no contribution to the angular momentum from an electromagnetic field. The only contribution is from the particles of the the current traveling around the loop. Using the relation $M = \frac{2m_q}{q}\mathbf{m}$ [14] and recognizing that the magnetic moment of a loop is $\mathbf{m} = j\pi r^2\ \hat{\boldsymbol{z}}$. The energy from the angular momentum is,

$$-\mathbf{\Omega}\cdot\mathbf{M} = -4\pi^2\frac{m_e j^2 r^2}{Nq^2}. \tag{6.5}$$

The interaction energy between the external magnetic field and the magnetic dipole of the loop is calculated to be,

$$\mathbf{m}\cdot\mathbf{B}_{\text{ext}} = -j\pi r^2 B_{\text{ext}} \tag{6.6}$$

To get the point across we can combine all these terms and write the free energy in the simple form,

$$E'_{\text{vorton}} = \alpha r - \beta r^2, \tag{6.7}$$

where α and β are just ugly constants. Extremizing the free energy we see that the system has a maximum when,

$$r_0 = \frac{\alpha}{2\beta(1 + B_{\text{ext}})}. \tag{6.8}$$

If the vorton is initially smaller than r_0 its radius will still go to zero but if its radius is larger than r_0 the loop will continue to grow until it hits the container that the superfluid rests in. In dimensionless form the loop will grow if

$$1 > \frac{\alpha}{2R\beta}\frac{1}{(1 + B_{\text{ext}})}\,, \tag{6.9}$$

where R is the size of the vorton. A sufficiantly large external magnetic field B_{ext} will satisfy this inequality and make the vorton stable. This possibility of stability is a key requirement for the existence of vortex sheets and would also allow toroidal flux to be present inside the star.

6.2 Interactions between vortons

We now want to investigate what would happen if a number of stable vortices existed in the superfluid and try to determine if these configurations have lower energy than canonical vortex solutions.

If a vorton can be stable in the superfluid it is reasonable to ask how the vortices might interact with each other. To gain insight on these interactions this section will look a the force between two stable loops carrying current. Remember that in the regime where vortices attract it is because the force due to the current is dominant. To calculate the force we will first calculate the mutual inductance for the loops,

$$\begin{aligned} M_{12} &= \frac{\mu_0}{4\pi}\oint\oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{R}|}\,, \\ &= \frac{\Phi_2}{j_1}\,, \end{aligned} \tag{6.10}$$

where the subscripts 1 and 2 refer to each loop and Φ_2 is the flux through loop 2 caused by the current j_1 from loop 1. Taking derivatives of M_{12} gives the force [14] between the loops,

$$\mathbf{F} = j_1 j_2 \nabla M_{12}(\mathbf{R})\,. \tag{6.11}$$

The magnetic field of a loop is difficult to calculate and it is easier to use the vector potential. Since the current in the wire only runs in the ϕ direction the vector potential only has a ϕ component,

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\frac{4j_1 a}{\sqrt{a^2 + r^2 + 2ar\sin(\theta)}}\left[\frac{(2 - k^2)K(k) - 2E(k)}{k^2}\right]\,, \tag{6.12}$$

where K and E are elliptic functions, a is the size of the loop, r is point you want to evaluate the field, θ is the angle between $\hat{\boldsymbol{z}}$ and $\mathbf{r}$, and

$$k^2 = \frac{4ar\sin\theta}{a^2 + r^2 + 2ar\sin\theta}\,. \tag{6.13}$$

A number calculations[1] can be can be carried out to show that two current loops will always attract to one another and will tend to stack, forming a solenoid.

It is not hard to extend these calculations if the loops are now vortices with current in them strong enough to attract each other. It is suspected that the force of repulsion from the order field and the attractive force of the current will balance such that the distance between them is of the order of the penetration depth ξ. A large number free vortons will tend to form columns that loops like vortex solenoids. Figure D.3 also shows that two loops will exhibit simple harmonic motion if perturbed. This means that the vortex solenoid could carry waves along its length as well.

6.3 The Energy of a System Made of Vortex Sheets

The vortex cylinders that are formed from attracting vortices are very similar to the vortex sheets proposed by Landau and Lifshitz. We will consider a situation very similar to theirs where a large number of concentric layers exist in a superfluid at radii r_1, r_2,...,r_n. The system has two main components contributing to its energy - the rotating superfluid and the current running along the core of the vortex.

The first component is the superfluid flowing between the vortex layers which carries a circulation b_i. The velocity of the superfluid is given by $v_i = \frac{b_i}{r}$ which means that the velocity is discontinuous at a vortex layer. The energy of the superfluid component is

$$E_{\rm f} = \sum_i \pi\rho_{\rm s} b_i^2 \ln\frac{r_{i+1}}{r_i} + 2\pi r_i \frac{\hbar^2\pi\rho_{\rm s}}{dm_{\rm n}^2}\ln\frac{\Lambda}{\xi}\,. \tag{6.14}$$

where the first term is the kinetic energy of the superfluid and the second term is the energy of the vortex sheet created by stacking vortex loops on top of each other a distance d apart. Only the energy of the vortex is considered here and not the energy of the current running along its core. If this system is in thermal equilibrium there is a minimum when

$$E'_{\rm f} = E_{\rm f} - M_{\rm f}\Omega\,, \tag{6.15}$$

[1]For the details of these calculations see appendix D.

where M is the angular momentum of the system. The angular momentum for an irrotatable fluid is given by

$$M_{\mathrm{f}} = \rho_{\mathrm{s}}\pi \sum_i b_i \left(r_{i+1}^2 - r_i^2\right) . \tag{6.16}$$

Substituting 6.14 and 6.16 into 6.15 gives the free energy

$$E' = \pi\rho_{\mathrm{s}} \sum_i \left[b_i^2 \ln \frac{r_{i+1}}{r_i} - \Omega b_i \left(r_{i+1}^2 - r_i^2\right)\right] + \sum_i 2\pi r_i \frac{\hbar^2 \pi \rho_{\mathrm{s}}}{d m_n^2} \ln \frac{\Lambda}{d} . \tag{6.17}$$

Minimizing this with respect to the circulation b_i yields

$$b_i = \frac{\Omega}{2} \frac{r_{i+1}^2 - r_i^2}{\ln \frac{r_{i+1}}{r_i}} . \tag{6.18}$$

Substituting this back into our expression for the energy gives us the free energy for the i^{th} layer of the superfluid component,

$$E_i' = -\frac{\pi\rho_{\mathrm{s}}\Omega^2}{4} \frac{\left(r_{i+1}^2 - r_i^2\right)^2}{ln \frac{r_{i+1}}{r_i}} + 2\pi r_i \frac{\hbar^2 \pi \rho_{\mathrm{s}}}{d m_{\mathrm{n}}^2} \ln \frac{\Lambda}{\xi} . \tag{6.19}$$

The second component contributing to the energy is that of the current inside the vortices. The vortex structure is similar to that of a solenoid which is carrying a surface current density $K = \frac{j}{d}$, where j is the current carried along the vortex and d is the spacing between vortices stacked on top of each other. We assume that the system we're working with has a neutral charge so the only electromagnetic contribution F_{EM} is from the magnetic field. There is a mechanical contribution from the kinetic energy of the particles of mass m which make up the current. And since there is only a magnetic field, the only contribution to the angular momentum comes from the particles moving in a circle. The energy we'd like to minimize is

$$E_{\mathrm{c}}' = E_{\mathrm{EM}} + E_{\mathrm{mech}} - M_{\mathrm{mech}}\Omega . \tag{6.20}$$

The first term E_{EM} is obtained by assuming the current structure mimics that of a solenoid of radius r_i where the magnetic field $B = \mu_0 K \hat{z}$ exists only inside r_i. Integrating the magnetic field gives the energy per unit length of the solenoid,

$$E_{\mathrm{EM}} = \frac{1}{2\mu_0} \int_0^{r_i} \left(\frac{\mu_0 j}{d}\right)^2 r dr d\theta = \frac{\mu_0 \pi}{2} \left(\frac{j r_i}{d}\right)^2 . \tag{6.21}$$

To write the second and third terms we need an expression for the velocity v in terms of the current. The current in a loop can be written as the total charge in the loop over the time it takes for one particle to travel around the loop $j = \frac{Nqv}{2\pi r_i}$, where N is the number of particles in the current loop, q is the charge per particle and r_i is the radius of the loop. For a bunch of loops stacked on top of each other with spacing d the energy per unit length is

$$E_{\mathrm{mech}} = \frac{1}{2} \frac{N m_{\mathrm{q}}}{d} v^2 = 2\pi^2 \frac{m_q j^2 r_i^2}{d N q^2} . \tag{6.22}$$

The relationship between the angular momentum that a current loop carries and its magnetic moment is $M = \frac{2m}{q}\mathbf{m}$, where $\mathbf{m}$ is the magnetic moment of the loop. If we consider a current sheet with a current density $K = \frac{j}{d}$ then the magnetic moment per unit length $\frac{j}{d}\pi r_i^2$. Then we can find the energy per unit length to be

$$-\Omega M_{\text{mech}} = -\frac{v}{r_i}\frac{2m_e}{q}\mathbf{m} = -4\pi^2\frac{m_{\text{q}}j^2r_i^2}{dNq^2}\,. \tag{6.23}$$

Substituting 6.21, 6.22 and 6.23 into 6.20 gives free energy of the i^{th} layer which comes from the current,

$$E_{\text{c}}' = \frac{\mu_0\pi}{2}\left(\frac{jr_i}{d}\right)^2 - 2\pi^2\frac{m_{\text{q}}j^2r_i^2}{dNq^2}\,. \tag{6.24}$$

It is also interesting to look at what happens when the system is immersed in an external magnetic field, B_{ext}. This magnetic field couples to the magnetic dipole of the solenoids creating a term which lowers the free energy of the system by

$$E_{\text{B}_{\text{ext}}} = -\pi jB_{ext}r_i^2\,. \tag{6.25}$$

There definitely exists an interaction between sheets which comes from the current. At this point we will assume that the interaction from the external field is much larger than this. In a neutron star this would be a reasonable assumption. If the currents formed after the formation of the star the magnetic field they produce would be suppressed by the Meisner effect and sheets would only interact significantly on the order of the penetration depth apart. The colossal magnetic field of a neutron star would overshadow the interaction otherwise.

Combining 6.19, 6.24 and 6.25 and rewriting it in terms of dimensionless parameters we get the total energy for our system. The first term in the expression below is associated with the rotational free energy, the second term is the surface energy of the vortex sheet, the third term is the energy energy of the magnetic field created by the current configuration and the fourth term is the energy from the angular momentum associated with the current,

$$E_i' = \frac{\pi}{2}\frac{\mu_0j^2R^2}{d^2}\left(-\alpha\frac{\left(x_{i+1}^2 - x_i^2\right)^2}{ln\frac{x_{i+1}}{x_i}} + \sigma x_i + (1-\eta-\beta)x_i^2\right)\,, \tag{6.26}$$

where,

$$x_i = \frac{r_i}{R},\;\; \alpha = \frac{\rho_{\text{s}}\Omega^2R^2d^2}{2\mu_0j^2},\;\; \sigma = \frac{\hbar^2\rho_{\text{s}}d}{\mu_0m_{\text{n}}^2j^2R}\ln\frac{\Lambda}{\xi}\,,$$
$$\eta = \frac{4\pi m_{\text{q}}d}{Nq^2}\,,\;\; \beta = \frac{2d}{\mu_0j}B_{\text{ext}}\,. \tag{6.27}$$

6.4 Minimizing the Free Energy

The lowest energy configuration for the vortex sheets is found by minimizing the energy with respect to the distance between sheets, h. Unlike the layered structure introduced by Landau and Lifshitz the layers of surface tension carry angular momentum via the current inside them.

Following Landau's example we will minimize in the regime where $\alpha \gg \sigma, \eta, 1$. Here sheets become evenly distributed throughout the liquid and are spaced at distances h small compared to R. In the limit of $h \to 0$ we should reproduce the energy of a normal liquid spinning as a solid body,

$$E_{\mathrm{s}}' = -\frac{\pi \rho \mathrm{s} \Omega^2}{4} \sum_i (r_{i+1}^4 - r_i^4) = -\frac{\mu_0 j^2 R^2}{d^2} \sum_i \alpha(x_{i+1}^4 - x_i^4) \,. \tag{6.28}$$

Subtracting 6.28 from 6.26 gives the energy contribution from just the sheets, $E_{\mathrm{sheet}} = E_i - E_{\mathrm{s}}$. If we define $\Delta = \frac{h}{R}$, substitute $x_{i+1} = x_i + \Delta$ into E_{sheet}, and expanding around in $\Delta = 0$ we get

$$E_{\mathrm{sheet}}' = \frac{\pi}{2} \frac{\mu_0 j^2 R^2}{d^2} \left(\alpha x_i + (1 - \eta - \beta) x_i^2 + \frac{4}{3} \alpha x_i \Delta^3 \right) . \tag{6.29}$$

Multiplying this by the number of sheets in the container, $\frac{R}{h} = \frac{1}{\Delta}$, gives us the energy for all the sheets and minimizing with respect to Δ yields the optimal sheet spacing,

$$\Delta = \left(\frac{3(\sigma + (1 - \eta - \beta) x_i)}{8\alpha} \right)^{1/3} . \tag{6.30}$$

Notice that Δ only gives a physical answer when $\sigma + (1 - \eta - \beta) x_i > 0$. This is not a true restriction parameters but a reflection of our starting setup for the system.

Substituting Δ back into the energy gives us the minimal free energy of just the sheets,

$$E_{\mathrm{sheet}}' = \frac{\pi}{2} \frac{\mu_0 j^2 R^2}{d^2} \left(\frac{3}{2} x_i (\alpha + (1 - \eta - \beta) x_i) \right) , \tag{6.31}$$

and we see that $\sigma + (1 - \eta - \beta) x_i < 0$ is where the system has a negative free energy.

6.5 Comparison between vortex sheet energy and vortex array energy

For a superfluid in which the vortices do not support currents running along their axes the most favorable configuration is an array of vortices described in chapter 3. In a superfluid where a current does run along the axis of the vortices this may no longer be true. In an array the vortices all form parallel to the axis of rotation which implies that the currents also run parallel to the axis of rotation. While arranged such the current adds energy to the system but does not can contribute to the angular momentum. A vortex sheet as proposed will carry a current in the plane of rotation where it can contribute to the angular momentum of the system.

To determine the regimes in which each configuration is favored we will compare the free energy of the vortex sheets E_{sheet} 6.26 with the free energy of a vortex array E_{array} 3.49 with a term added to account for the current in each vortex,

$$E'_{\text{array}} = -\frac{\pi}{4}\rho s R^4 \Omega^2 + \frac{1}{2}\rho \Gamma_0 R^2 \Omega \left(\ln\left(\frac{\Lambda}{\xi}\right) - \frac{1}{2} \right) + N_v \left(\frac{\mu_0 j^2}{16\pi}\right), \tag{6.32}$$

where Γ_0 is the quanta of circulation for the fluid. The last term is the energy per unit length of a current multiplied by the number of vortices. Using the Feynman-Onsager formula 3.40 the number of vortices can be written as $N_{\text{v}} = \frac{2\pi R^2 \Omega}{\Gamma_0}$. For the comparison we take a cylindrical slice of E_{array} between x_i and x_{i+1} which corresponds to a single term in E_{sheet},

$$E'_{\text{array}} = \frac{\pi}{2}\frac{\mu_0 j^2 R^2}{d^2}\left(-\alpha(x_{i+1}^4 - x_i^4) + \gamma(x_{i+1}^2 - x_i^2)\right), \tag{6.33}$$

where α is defined as before and

$$\gamma = d^2 \left(\frac{\rho \Gamma_0 \Omega}{\pi \mu_0 j^2} (\ln\left(\frac{\Lambda}{\xi}\right) - \frac{1}{2}) + \frac{\Omega}{4\pi\Gamma_0} \right). \tag{6.34}$$

Comparing these two cylindrical slices of width Δ will be sufficient to compare the entire energy. When the quantity $\delta E = E'_{\text{array}} - E'_{\text{sheet}}$ is positive the vortex array solution is favored and when it is negative the vortex sheet solution is favored. Substituting $x_{i+1} = x_i + \Delta$ into δE and expanding around Δ gives

$$\delta E = \frac{\pi}{2}\frac{\mu_0 j^2 R^2}{d^2}\left((\sigma + (1-\eta-\beta)x_i)x_i - 2\gamma x_i \Delta - \gamma\Delta^2 + \frac{4}{9}\alpha x_i \Delta^3 \right). \tag{6.35}$$

If we assume $\Delta \ll 1$ then we only use up to the first order in Δ and we get

$$\delta E = \frac{\pi}{2}\frac{\mu_0 j^2 R^2}{d^2}\left((\sigma + (1-\eta-\beta)x_i)x_i - 2\gamma x_i \left(\frac{3(\sigma + (1-\eta-\beta)x_i)}{8\alpha}\right)^{1/3} \right). \tag{6.36}$$

If this is treated as a polynomial in $A = \sigma + (1-\eta-\beta)x_i$ then it has roots when

$$A = 0,\ \pm\sqrt{\frac{3\gamma^3}{\alpha}}. \tag{6.37}$$

Using these roots we can see that $\delta E < 0$ when,

$$0 < \sigma + (1-\eta-\beta)x_i < \sqrt{\frac{3\gamma^3}{\alpha}}, \tag{6.38}$$

the vortex sheet is a more favorable configuration energetically than an array of vortex lines. This is the only physically viable solution because the spacing between layers becomes complex when $\sigma + (1-\eta-\beta)x_i < 0$. In implies there is a dramatic change in the system and our initial configuration no longer describes it.

6.6 Application to Neutron Stars

The inequality 6.38 is quite general. The only real requirement is that there is a current which exits along with a superfluid. The only requirement for there to be a current is there exist modes which are described by a massless Dirac theory. These massless modes automatically imply a current is present.

We will apply this inequality to the neutron superfluid in a neutron star to see if, and when, the vortex sheet will be favored over the canonical vortex structure. It is assumed here that the neutrons form the superfluid and that zero modes in the proton superconductor are responsible for the current in the axes of the neutron vortices. The number of particles that make up the current in a single vortex loop can be calculated using

$$N = \frac{\rho_{\mathrm{q}}}{m_{\mathrm{q}}} 2\pi^3 R x_i \xi^2 = 2\pi R j \, . \tag{6.39}$$

The spacing between the vortex loops which make up the vortex sheet is taken to be on the order of the coherence length of the superfluid. This is where the attractive force from the current and the repulsive force from the order field would balance out. The value of the current is taken from the calculation by Zhitnitsky and Son. Values for physical parameters in a neutron star are presented in the following table.

Table 6.1: Physical constants in a neutron star in SI units and "God given" units where $\hbar = c = \mu_0 = 1$.

Variable	Description	SI	God
R	Radius of the neutron star	10^4 m	$(10^{-11}\mathrm{eV})^{-1}$
Ω	Angular velocity of the neutron star	10^2 s^{-1}	10^{-12} eV
ρ_{s}	Superfluid density	10^{18} kg/m^3	$(100\ \mathrm{MeV})^4$
m_{n}	Superfluid particle mass	10^{-27} kg	939 MeV
ξ	Coherence length of superfluid	10^{-14} m	$(10\ \mathrm{MeV})^{-1}$
d	Distance between superfluid vortices	10^{-14} m	$(10\ \mathrm{MeV})^{-1}$
ρ_{q}	Current particle density	10^{18} kg/m^3	$(100\ \mathrm{MeV})^4$
m_{q}	Current particle mass	10^{-27} kg	939 MeV
q	Current particle charge	10^{-19} C	$\sqrt{\frac{4\pi}{137}}$
j	Current	10^2 A	10 MeV
N	Number of particles in the current	10^{18}	10^{18}

Substituting these values into the dimensionless parameters 6.27 gives their intrinsic values listed the table below.

These can be used to evaluate each side of the inequality 6.38,

$$0 < \sigma + (1 - \eta - \beta) x_i < \sqrt{\frac{3\gamma^3}{\alpha}} \, , \tag{6.40}$$

in two different cases; one with the external magnetic field, B_{ext}, set to zero and the other with an eternal field. In the first case the inequality approximately

Table 6.2: Dimensionless parameters and their values.

Symbol	Parameter	Value
α	$\frac{\rho_s \Omega^2 R^2 d^2}{2\mu_0 j^2}$	10
η	$\frac{4\pi m_q d}{N q^2}$	10^{-15}
σ	$\frac{\hbar^2 \rho_s d}{\mu_0 m_n^2 j^2 R} \ln \frac{\Lambda}{\xi}$	10^{-18}
γ	$d^2 \left(\frac{\rho \Gamma_0 \Omega}{\pi \mu_0 j^2} (\ln \left(\frac{\Lambda}{\xi} \right) - \frac{1}{2}) + \frac{\Omega}{4\pi\Gamma_0} \right)$	$10^{-17} + 10^{-20}$

becomes,

$$1 < 0 \,. \tag{6.41}$$

The inequality is quite clearly violated meaning that the canonical vortex array is more energetically favorable than the vortex sheets. Physically, it is the energy of the creating the magnetic field inside the vortex sheet which is responsible for the 1 on the left hand side. This means that producing the field is much more costly than having currents running along the axis of vortices in an array, contributing to the energy, but not to the angular momentum.

The next step is the introduce an external magnetic field. Doing this the inequality becomes,

$$1 < \beta \Longrightarrow B_{\text{ext}} > \frac{\mu_0 j}{2d} \sim 10^{11}\ \text{T} = 10^{15}\ \text{G} \,. \tag{6.42}$$

This implies that the solutions is only favorable for for extremely large magnetic fields. Fields this size are only found in magnetars, neutrons stars with abnormally large magnetic fields of 10^{15} G.

An inconsistency in the equations arises here in the assumption that the external magnetic field contributes to the interaction between sheets much more than the magnetic field produced by the sheets. A single vortex sheet, regardless of its radius, produces a magnetic field

$$B = \frac{\mu_0 j}{d} \sim 10^{12}\ \text{T} = 10^{16}\ \text{G} \,. \tag{6.43}$$

This means that the interactions between the sheets, which carry current loops, and the magnetic fields produced by them might start to become important, important enough that the field required for the vortex sheets to be stable is dramatically reduced. It is interesting to note that the field produced from the vortex sheet is of the order of fields found in magnetars.

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Appendix A

A complicated way to calculate the force between two wires carrying current

There are a number of ways to calculate the force between two wires carrying current. They invariably use the Lorentz Force Law which requires you to find the magnetic field. Some ways are easy, such as using Ampere's Law, some are harder, such as using the Biot-Savart Law, and then there are some you would never do. It is one of the latter which is presented here. I used it as a quaint demonstration that the technique for finding the interaction force between two vortices actually made sense and, because it is quaint and mildly useful, I believe it should be reproduced for everyone to see, similar to how one would paint a water-colour of a cabin on a lake-front and give it to their friends.

Consider the energy of a single wire, carrying curent, running along the z-axis. The energy per unit length of this system is given by,

$$E = \int dx^2 \left(\frac{1}{\mu_0} (\nabla \times \mathbf{A})^2 + \mathbf{j} \cdot \mathbf{A} \right) \tag{A.1}$$

where $\mathbf{j} = (0, 0, j\delta(r))$. The equation of motion for the vector field is,

$$\nabla \times \nabla \times \mathbf{A} = -\mu_0 \mathbf{j} \tag{A.2}$$

Which is just Ampere's Law in differential form. Instead of transforming it to the integral form we will mimic the vortex method and solve the differential equation directly. Because of the cylidrical symetry of our problem it is convenient to solve this in cylindrical coordinates. Using the anstaz $\mathbf{A} = (0, 0, A_z(r))$ we write it as,

$$\nabla^2 A_z(r) = j\delta(r)\,. \tag{A.3}$$

To find a solution for this equation we will consider the object $\nabla \times \nabla \times \ln r\ \hat{\boldsymbol{z}} = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \ln r$. On first glance this is just zero but things get interesting when we take the surface integral of this.

$$\int_S (\nabla \times \nabla \times \ln r\ \hat{\boldsymbol{z}}) \cdot d\mathbf{a} = \oint (\nabla \times \ln r\ \hat{\boldsymbol{z}}) \cdot d\mathbf{l} = -\oint \frac{1}{r} r d\phi = -2\pi \tag{A.4}$$

This apparent paradox means means we've discovered a delta function in our midst and that

$$\nabla \times \nabla \times \ln r\ \hat{\boldsymbol{z}} = -2\pi\delta(r) \tag{A.5}$$

Comapring this with our differential equation gives the solution,

$$A_z = \frac{\mu_0 j}{2\pi} \ln r \tag{A.6}$$

The interaction energy is given by

$$\begin{aligned}
E_{\text{int}} &= \int dx^2 \left[E(r_1 + r_2) - E(r_1) - E(r_2)\right] &\text{(A.7)}\\
&= \int dx^2 \left[\frac{2}{\mu_0}(\nabla \times \mathbf{A}_1)(\nabla \times \mathbf{A}_2) + j_1 A_2 + j_2 A_1\right] &\text{(A.8)}\\
&= \int dx^2 \mathbf{j}_1 \mathbf{A}_2 &\text{(A.9)}\\
&= \int dx^2 j_1 \delta(x - x_1) \frac{\mu_0 j_2}{2\pi} \ln(x - x_2) &\text{(A.10)}\\
&= \int \frac{\mu_0 j_1 j_2}{2\pi} \ln(x_1 - x_2) &\text{(A.11)}
\end{aligned}$$

The force is defined as,

$$F = -\frac{\partial E_{\text{int}}}{\partial d} = -\frac{\mu_0 j_1 j_2}{2\pi d} \tag{A.12}$$

where $d = |x_1 - x_2|$. The is the expression for the force between the two wires, which is what we expect to get. If the two currents travel in the same direction the force is negative and the two wires attract to one another and if the two currents travel in opposite directions the force is positive and the two wires find eachother repulsive.

Appendix B

Rotating Superconductors

Much of this thesis has been spent exploring the consequences of rotating a superfluid. Because it is also a condensate one might expect that a rotating superconductor would form an array of vortices carrying quantized circulation, but it does not. This question is critical for the observations of a neutron star and for many of the theories on glitches because it is the fact that a superconductor couples to the crust, and consequantly it's flux tubes, that allows us to determine the rotation of a neutron star. Becuase the superconductor is responsible for the magnetic field if it did form vortices it would be subject to all the metastable flows of a superfluid and glitch events how we imagined them would stil made the crust travel faster but not the magnetic field and we would never observe them.

The difference is that in descibing a superconductor we require gauge invariance which means that the velocity operator must be guage invariant. It carries a covariant derivative and when it acts on the wave function we get a velocity,

$$\mathbf{v} = -i\frac{\hbar}{m}\nabla\phi - \frac{q}{mc}\mathbf{A} \tag{B.1}$$

This result can also be obtained directly from the Noether Current 2.4 by substituting in the wave function and recognising that $j = n_o v$ as in equation 3.12. The presence of the gauge field removes the restriction on the velocity field. The fluid can now carry vorticity,

$$\nabla \times \mathbf{v} = -\frac{q}{mc}\nabla \times \mathbf{A} \tag{B.2}$$

and can rotate like a solid body. The vorticity of a solid body is 2Ω. Equating this with the vorticy derived for a superconductor yields,

$$\mathbf{B}_{\text{London}} = -\frac{mc}{q}\mathbf{\Omega} \tag{B.3}$$

The superconductor corotates with the container at the expense of small magnetic field called the London Field.

Appendix C

Interaction between superfluid vortices carrying current

The calculation for interactions between superfluid vortices is quite similar to the calculation for superconducting vortices. The calculation could be reproduced, but as seen in chapter 5 adding a current does not change any of the original interaction terms calculated in 3.28, it just adds a new one. The interaction for superfluid vortices carrying current is then,

$$E_{\text{int}} = \left(\frac{\hbar c}{q\lambda}\right)^2 \frac{1}{\sqrt{2}\xi} K_0\left(\frac{\sqrt{2}}{\xi}d\right) - 2j_1 j_2 K_0\left(\frac{d}{\lambda}\right), \tag{C.1}$$

and we get a situation very similar to the vortex interactions for a gauged field without current 3.28. The difference is that the sign of the interaction is reversed. If the coefficients in for each of the the terms above are the same order of magnitude then we get a Landau-Ginzburg parameter which says that two vortices repel when

$$\kappa < \frac{1}{\sqrt{2}} \quad \text{where} \quad \kappa = \frac{\lambda}{\xi}. \tag{C.2}$$

Comparing this with 2.52 we see that the condition is reversed.

Appendix D

Interaction between two current loops

To calculate the force between two current loops we will first calculate the mutual inductance for the loops,

$$\begin{aligned} M_{12} &= \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{R}|} \,, \\ &= \frac{\Phi_2}{j_1} \,, \end{aligned} \tag{D.1}$$

where the subscripts 1 and 2 refer to each loop and Φ_2 is the flux through loop 2 caused by the current j_1 from loop 1. Taking derivatives of M_{12} gives the force between the loops,

$$\mathbf{F} = j_1 j_2 \nabla M_{12}(\mathbf{R}) \,. \tag{D.2}$$

The magnetic field of a loop is difficult to calculate and it is easier to use the vector potential. Since the current in the wire only runs in the ϕ direction the vector potential only has a ϕ component,

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi} \frac{4 j_1 a}{\sqrt{a^2 + r^2 + 2ar\sin(\theta)}} \left[\frac{(2-k^2)K(k) - 2E(k)}{k^2} \right] , \tag{D.3}$$

where K and E are elliptic functions, a is the size of the loop, r is point you want to evaluate the field, θ is the angle between $\hat{\boldsymbol{z}}$ and $\mathbf{r}$, and

$$k^2 = \frac{4ar\sin\theta}{a^2 + r^2 + 2ar\sin\theta} \,. \tag{D.4}$$

Consider the case where two loops of radius a and b where $a \approx b$ which are located a distance d apart on a common axis perpendicular to their planes. Placing loop 2 so its center rests on the origin, the flux though it caused by 1 can be written in terms of a path integral using Stokes theorem,

$$\Phi_2 = \int_S \mathbf{B_1} \cdot d\mathbf{a} = \int_S \nabla \times \mathbf{A_1} \cdot d\mathbf{a} = \int_{\partial S} \mathbf{A_1} \cdot d\mathbf{l} \tag{D.5}$$

$$= \int_0^{2\pi} A_1 b d\phi = 2\pi b A(b) \,. \tag{D.6}$$

The vector potential from loop 1 evaluated on the plane in which loop 2 rests is given by substituting $r = \sqrt{d^2 + x^2}$, where x is the distance from the origin on the x-y plane. Also $\sin(\theta) = \frac{x}{r}$ so $k^2 = \frac{4ax}{(a^2+x^2)^2+d^2}$. The magnetic induction is then

$$M_{12} = \mu_0 \sqrt{ab} \left[\frac{(2-k^2)K(k) - 2E(k)}{k} \right] , \tag{D.7}$$

where,

$$k^2 = \frac{4ab}{(a+b)^2 + d^2} . \tag{D.8}$$

Taking the derivative of this with respect to d and plotting it with $a = b = 1$ gives figure D.1. We can clearly see that the force is always attractive (negative) and there is no force when the loops lie on top of each other.

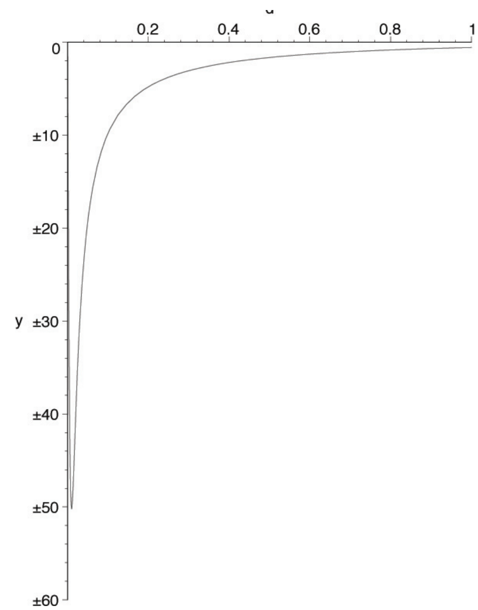

Figure D.1: The force between two current loops on a common axis perpendicular to their planes as a function of the distance d between them.

The second case is when $d \sim 0$ and $a \neq b$. We can use the same mutual inductance derived for the first case but here we are interested in the force on the radii of the loops. Setting $a = 1$ and taking the derivative of equation D.7 with respect to b we get the force on loop 2 if the radius of loop 1 is fixed at $a = 1$. The force is shown in figure D.2 and it can see that b is always drawn to $a = 1$ and that loop loops want to rest on each other.

The third case is when two loops of radius a rest in the same plane but their centres are displaced a distance c apart. We want to evaluate the line integral

$$\int \mathbf{A}_1 \cdot d\mathbf{l} = \int \mathbf{A}_1(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt . \tag{D.9}$$

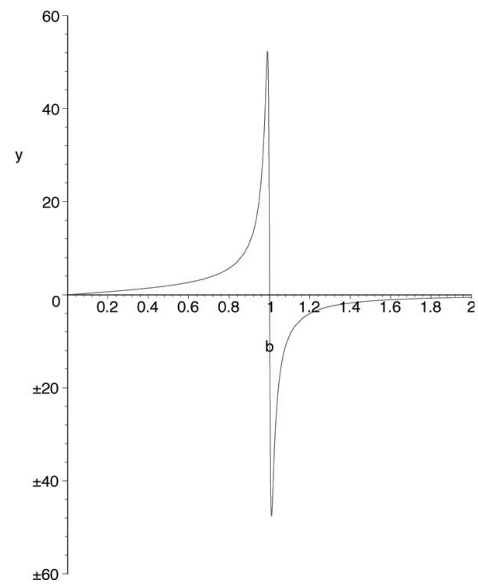

Figure D.2: The force between two loops on a common axis perpendicular to their planes as a function of the radius of one b changing with respect to the other $a = 1$.

For two loops separated such that c is in the $\hat{\boldsymbol{x}}$ direction, $\mathbf{r} = (a\cos t + c)\ \hat{\boldsymbol{x}} + a\sin t\ \hat{\boldsymbol{y}}$ for $t = 0, 2\pi$. The magnitude of $\mathbf{r}$ is $r = \sqrt{a^2 + c^2 + 2ac\cos t}$ and $\mathbf{r}' = -a\sin t\ \hat{\boldsymbol{x}} + a\cos t\ \hat{\boldsymbol{y}}$.

To take the dot product of the vector field it is easier to change from spherical to cartesian coordinates and write ϕ in terms of t;

$$\begin{aligned} \hat{\boldsymbol{\phi}} &= -\sin(\phi)\ \hat{\boldsymbol{x}} + \cos(\phi)\ \hat{\boldsymbol{y}} & \text{(D.10)} \\ &= \frac{-a\sin t}{r}\ \hat{\boldsymbol{x}} + \frac{a\cos t + c}{r}\ \hat{\boldsymbol{y}}\,. & \text{(D.11)} \end{aligned}$$

The integral then becomes

$$\int_0^{2\pi} A(k(t)) \frac{a^2 + ac\cos t}{\sqrt{a^2 + c^2 + 2ac\cos t}} dt\,. \tag{D.12}$$

Because $\mathbf{r}$ rests in the xy-plane $\sin\theta = 1$ and

$$k(t)^2 = \frac{4a\sqrt{a^2 + c^2 + 2ac\cos t}}{2a^2 + b^2 + 2ab\cos t + 2a\sqrt{a^2 + c^2 + 2ac\cos t}}\,. \tag{D.13}$$

Taking the derivative of this integral with respect to c gives a integral over the force per unit length of wire. Figure D.3 shows the numerical solution of the integral. The loops will attract until the loops are almost no longer overlapping and they repel.

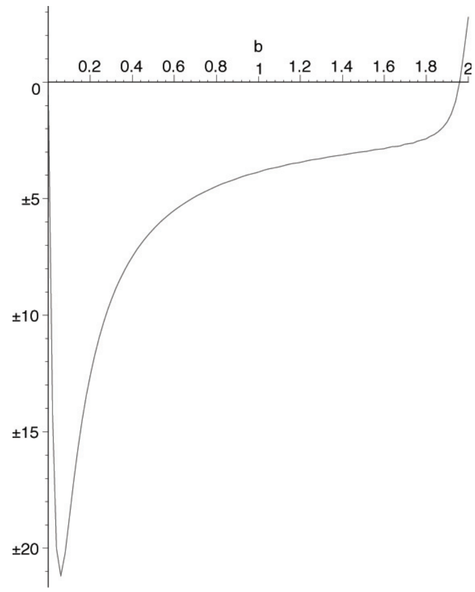

Figure D.3: The force between two loops displaced a distance b apart along a direction in their plane.

These calculations give a complete picture of the force between current loops carrying current.