8 Thompson and Compton scattering

An electromagnetic wave impinging on a charged particle, such as an electron, creates an oscillating motion of the charge. In turn, the oscillating charge generates radiation. This process is known as scattering. If the motion of the charge is nonrelativistic, the process is called Thompson scattering. The relativistic case is called Compton scattering.

8.1 Thompson scattering

Consider a linearly-polarized monochromatic plane wave incident on a particle of charge q and mass m initially at rest. The electric field at the particle has the form

$$\boldsymbol{E} = \mathrm{Re}[\boldsymbol{\mathcal{E}} e^{i\omega t}] = \boldsymbol{\mathcal{E}} \cos(\omega t). \tag{8.1}$$

The resulting Lorentz three-force on the particle is

$$\boldsymbol{f} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{8.2}$$

The second term can be neglected since $v << 1$ and $B = E$ in the wave. Thus, the resulting three-acceleration is

$$\boldsymbol{a} = \frac{\boldsymbol{f}}{m} = \frac{q\boldsymbol{\mathcal{E}}}{m} \cos(\omega t). \tag{8.3}$$

Putting this into Larmor's formulae (7.17) and (7.18) and taking the time average, we get

$$\frac{dP}{d\Omega} = \frac{q^4 \mathcal{E}^2}{32\pi^2 m^2} \sin^2 \varphi, \tag{8.4}$$

$$P = \frac{q^4 \mathcal{E}^2}{12\pi m^2} \tag{8.5}$$

The incident flux of the wave is given by the time average of the Pointing vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}$. Since the electric and magnetic fields are perpendicular, and have equal amplitudes,

$$F = \frac{1}{2}\mathcal{E}^2 \tag{8.6}$$

Define the *differential cross section* for scattering into angle φ by

$$\frac{d\sigma}{d\Omega} = \frac{dP}{F d\Omega}. \tag{8.7}$$

Therefore, for electron scattering we find

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{e^4}{16\pi^2 m^2} \sin^2 \varphi, \\ &= r_0^2 \sin^2 \varphi, \end{aligned} \tag{8.8}$$

where

$$r_0 = \frac{e^2}{4\pi\varepsilon_0 mc^2} \tag{8.9}$$

is the *classical electron radius.*

Integrating over solid angle gives the total cross section

$$\sigma = \sigma_T \equiv \frac{8\pi}{3} r_0^2, \tag{8.10}$$

which is called the *Thompson cross section.*

The differential cross section for unpolarized radiation can be found by averaging around the direction of the incident radiation. Drawing a spherical triangle with vertices corresponding to the directions of the incident and outgoing waves and the electric field vector, one finds $\cos\varphi = \sin\theta\cos\phi$, so

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= r_0^2(1 - \langle\cos^2\phi\rangle\sin^2\theta), \\ &= r_0^2(1 - \frac{1}{2}\sin^2\theta), \\ &= \frac{1}{2}r_0^2(1 + \cos^2\theta). \end{aligned} \tag{8.11}$$

In the rest frame of the particle, the incident and scattered radiation has the same frequency. Therefore, the energy of an incident and scattered photon is the same. This is an example of *coherent scattering.*

8.2 Compton scattering

Compton scattering occurs when the energy of the incident photon is sufficiently great that significant momentum is imparted to the charged particle. As a result, the energy of the photon is changed by the scattering process. Let $\vec{k}_i$ and $\vec{k}_f$ be the initial and final four-frequencies of the photon. Similarly, let $\vec{p}_i$ and $\vec{p}_f$ be the initial and final four-momenta of the particle. (The subscripts here are labels, not vector indices). Then conservation of four-momentum requires that

$$\vec{k}_i + \vec{p}_i = \vec{k}_f + \vec{p}_f. \tag{8.12}$$

Chose a frame in which the particle is initially at rest. Then, $\vec{p}_i = m(1, \mathbf{0})$. The photon momenta are $\vec{k}_i = \omega_i(1, \boldsymbol{n}_i)$ and $\vec{k}_f = \omega_f(1, \boldsymbol{n}_f)$, where $\boldsymbol{n}_i$ and $\boldsymbol{n}_f$ are the initial and final directions of the photons ($\hbar = 1$). Then, we have

$$\begin{aligned} m^2 = p_f^2 &= (\vec{k}_i + \vec{p}_i - \vec{k}_f)^2, \\ &= m^2 + 2\vec{p}_i\cdot(\vec{k}_i - \vec{k}_f) - 2\vec{k}_i\cdot\vec{k}_f, \\ &= m^2 + 2m(\omega_i - \omega_f) - 2\omega_i\omega_f(1 - \boldsymbol{n}_i\cdot\boldsymbol{n}_f). \end{aligned} \tag{8.13}$$

In terms of the wavelength, $\lambda = 2\pi/\omega$, this becomes

$$\lambda_f = \lambda_i + \lambda_c(1 - \cos\varphi), \tag{8.14}$$

where φ is the angle between the initial and final photon direction and $\lambda_c = 2\pi/m = h/mc$ is the *Compton wavelength*. It is the wavelength for which $\hbar\omega = mc^2$. For an electron, $\lambda_c \sim 0.002426$ nm. Photons that have a wavelength much larger than this cannot change appreciably change the energy of the electron, so the collision corresponds to Thompson scattering. On the other hand, high-energy photons, with $\lambda << \lambda_c$ can accelerate the electron to relativistic velocity.

The cross section for Compton scattering is given by the Klein-Nishina formula, derived using quantum electrodynamics,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_0^2 \frac{\omega_f^2}{\omega_i^2} \left(\frac{\omega_i}{\omega_f} + \frac{\omega_f}{\omega_i} - \sin^2\varphi \right) \tag{8.15}$$

This is smaller than the for Thompson scattering. Scattering is less efficient at high energies.

The total scattering cross section is

$$\sigma = \sigma_T \frac{3}{4} \left\{ \frac{1+x}{x^3} \left[\frac{2x(1+x)}{1+2x} - \ln(1+2x) \right] + \frac{1}{2x} \ln(1+2x) - \frac{1+3x}{(1+2x)^2} \right\}, \tag{8.16}$$

where $x = \omega_i/m = \lambda_c/\lambda_i$. This is plotted for a range of x in Figure (8.1).

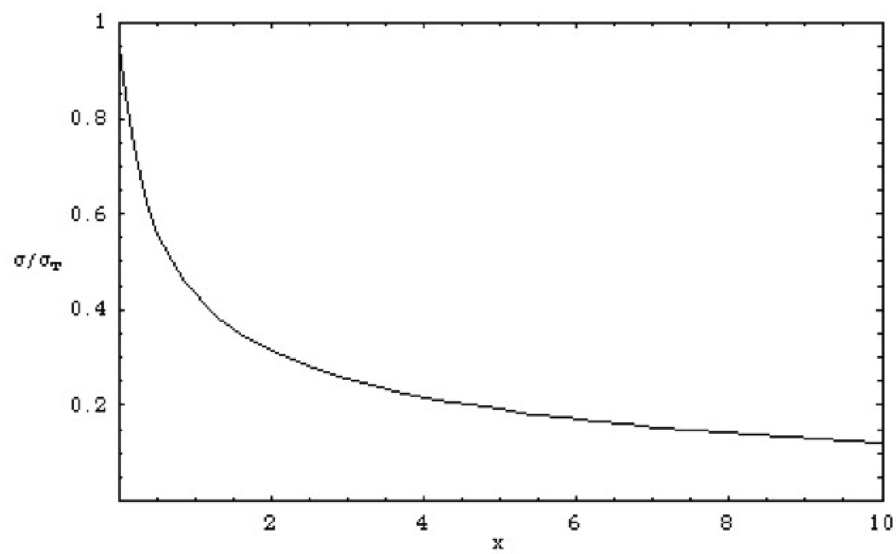

Figure 8.1: Compton scattering cross section. The figure shows the cross section, in units of the Thompson cross section, as a function of the dimensionless energy parameter $x = \omega_i/m = \lambda_c/\lambda_i$.