12 Bremsstrahlung

Bremsstrahlung refers to radiation that is produce when a moving charge is accelerated by the coulomb field of another charge. Most commonly this occurs in an astrophysical plasma when electrons are deflected as they pass near to ions. Electrons dominate the emission because their lower mass results in greater acceleration. Bremsstrahlung is also called free-free emission, as the electron is not in a bound state.

A complete analysis requires quantum electrodynamics. However, the basic results can be obtained from a classical analysis, to which are added corrections from the quantum theory.

12.1 Single non-relativistic electron

Consider an electron encountering an ion (Figure 12.1). Classically, the Coulomb attraction results in a deflection of the electron, which results in radiation. Ions are several thousand times more massive than an electron, so the ion can be considered as fixed. The deflection of the electron is generally very small so the path can be approximated by a straight line (for an exact approach, see Landau and Lifshitz, *The Classical Theory of Fields*).

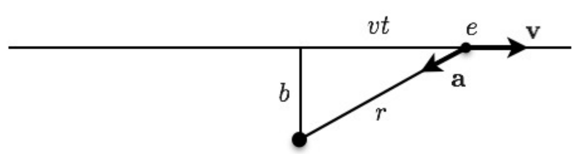

Figure 12.1: Geometry of electron-ion encounter.

Let the charge of the ion be Ze and let $\boldsymbol{r}$ be the vector connecting the ion to the electron. The acceleration of the electron is given by

$$\boldsymbol{a} = -\frac{Ze^2}{4\pi m r^3}\boldsymbol{r} = -\frac{Zr_0}{r^3}\boldsymbol{r} \tag{12.1}$$

Therefore,

$$a^2 = \frac{Z^2 r_0^2}{(b^2+v^2t^2)^2}. \tag{12.2}$$

Putting this into Larmor's formula (7.18) we get the total power radiated as a function of time,

$$P(t) = \frac{e^2a^2}{6\pi} = \frac{\alpha\sigma_T Z^2}{4\pi(b^2+v^2t^2)^2}., \tag{12.3}$$

where $\alpha = e^2/4\pi \simeq 1/137$ is the fine structure constant.

The total energy radiated is

$$\begin{aligned} W &= \int_{-\infty}^{\infty} P dt = \frac{\alpha\sigma_T Z^2}{4\pi}\int_{-\infty}^{\infty}\frac{dt}{(b^2+v^2t^2)^2}, \\ &= \frac{\alpha\sigma_T Z^2}{4vb^3}. \end{aligned} \tag{12.4}$$

12.2 Spectrum of the radiation

The spectrum of the emitted power is related to the Fourier transform of the acceleration of the electron, which we define by

$$\tilde{a}(\omega) = \int_{-\infty}^{\infty} a(t)e^{-i\omega t}dt, \tag{12.5}$$

$$a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{a}(\omega)e^{i\omega t}d\omega, \tag{12.6}$$

Using (12.3), we can write the radiated energy as

$$W = \int_{-\infty}^{\infty} P(t)dt = \frac{e^2}{6\,\pi}\int_{-\infty}^{\infty} a^2(t)dt. \tag{12.7}$$

By Parseval's theorem, this can be written as an integral over frequency,

$$W = \frac{e^2}{12\pi^2}\int_{-\infty}^{\infty} |\tilde{a}(\omega)|^2 d\omega. \tag{12.8}$$

Since $a(t)$ is real, $\tilde{a}(\omega)$ is Hermetian (its real part is symmetric and its imaginary part is antisymmetric). Thus

$$W = \frac{e^2}{6\pi^2}\int_{0}^{\infty} |\tilde{a}(\omega)|^2 d\omega. \tag{12.9}$$

From this we see that the energy emitted per unit angular frequency is given by

$$W_\omega \equiv \frac{dW}{d\omega} = \frac{e^2}{6\pi^2}|\tilde{a}(\omega)|^2. \tag{12.10}$$

Evaluating the Fourier transform, we get

$$\begin{aligned} \tilde{a}(\omega) &= Zr_0\int_{-\infty}^{\infty} \frac{e^{-i\omega t}dt}{b^2+v^2t^2}, \\ &= \frac{Zr_0}{vb}\int_{-\infty}^{\infty} \frac{e^{-i(\omega b/v)x}dx}{1+x^2}, \\ &= \frac{\pi Z r_0}{vb}e^{-|\omega b/v|}. \end{aligned} \tag{12.11}$$

Thus,

$$W_\omega = \frac{\alpha\sigma_T Z^2}{4v^2b^2}e^{-2\omega b/v}. \tag{12.12}$$

We see that the spectrum of the emitted power is essentially constant when $\omega \ll v/b$.

12.3 Emission from many electrons

Now suppose that we have many electrons and ions, with number densities n_e and n_i respectively, with the electrons all having the same speed v. The number of collisions per unit volume, per unit time, with impact parameter in the range $(b, b+db)$ is

$$\frac{dN}{dV\,dt} = n_e n_i v \cdot 2\pi b db. \tag{12.13}$$

Therefore, the energy emitted per unit frequency, per unit volume, is

$$\frac{dW}{dV\,d\omega dt} = \frac{\pi}{2v}\alpha\sigma_T Z^2 n_i n_e \int_{b_{min}}^{b_{max}} \frac{e^{-2\omega b/v}}{b} db, \tag{12.14}$$

where b_{min} and b_{max} are minimum and maximum impact parameters. As a good approximation we can take $b_{max} = \infty$, however as $b_{min} \to 0$ the power diverges logarithmically. A reasonable lower limit is the distance at which quantum effects become important. From the Heisenberg uncertainty principle, $\Delta p \Delta x \sim \pi$. Taking $\Delta x = b_{min}$ and $\Delta p = mv$ gives $b_{min} \sim \pi/mv$. Inserting this into (12.15) we get

$$\frac{dW}{dV\,d\omega dt} = \frac{\pi\alpha\sigma_T}{2v} Z^2 n_i n_e E_1(x),,$$

where $E_1(x)$ is an exponential integral and $x = 2\pi\omega/mv^2$.

The exact result, from quantum electrodynamics, is quite similar,

$$\frac{dW}{dV\,d\omega dt} = \frac{2\alpha\sigma_T}{\sqrt{3}\,v} Z^2 n_i n_e g_{ff}(v,\omega), \tag{12.15}$$

where g_{ff} is a slowly-varying function of v and ω called the *Gaunt factor*.

12.4 Thermal bremsstrahlung

Finally, we now allow a range of electron velocities. In astrophysics settings, the electrons typically have a thermal velocity distribution, given by the *Maxwell-Boltzmann distribution.* The probability the electron has a velocity $\boldsymbol{v}$, within d^3v is

$$f(\boldsymbol{v})d^3v = \left(\frac{m}{2\pi T}\right)^{3/2} e^{-mv^2/2T} d^3v \tag{12.16}$$

(The normalization constant comes from the condition that the integral of $P(\boldsymbol{v})$ over the entire three-velocity space must be unity.) Therefore,

$$f(v)dv = \left(\frac{m}{2\pi T}\right)^{3/2} 4\pi v^2 \exp(-mv^2/2k)dv. \tag{12.17}$$

Averaging the velocity over this distribution,

$$\frac{dW}{dV\,d\omega dt} = \frac{8\pi\alpha\sigma_T}{\sqrt{3}} \left(\frac{m}{2\pi T}\right)^{3/2} Z^2 n_i n_e \int_{v_{min}}^{\infty} g_{ff} e^{-mv^2/2T} v dv. \tag{12.18}$$

where $v_{min} = \sqrt{2\omega/m}$ is the minimum velocity of an electron that has at least energy ω. Setting $x = \sqrt{mv^2/2T}$ this can be written as

$$\frac{dW}{dV d\omega dt} = \frac{4m\alpha\sigma_T}{(6\pi mT)^{1/2}} Z^2 n_i n_e \, \bar{g}_{ff} e^{-\omega/T}, \tag{12.19}$$

where

$$\bar{g}_{ff} = \int_{x_{min}}^{\infty} g_{ff}(x) e^{-x} x dx \bigg/ \int_{x_{min}}^{\infty} e^{-x} x dx \tag{12.20}$$

is the Gaunt factor averaged over the velocity distribution. Its value may be approximated by the following equation, in SI units,

$$\bar{g}_{ff}(T,\nu) = 14.18 + 1.91\log(T) - 1.27\log(\nu). \tag{12.21}$$

If we assume isotropic emission and divide (12.19) by 4π steradians, we get the emission coefficient

$$\begin{aligned} j_\nu &= \frac{dW}{dV d\nu dt d\Omega} = \frac{2\pi}{4\pi}\frac{dW}{dV d\omega dt}, \\ &= 2\alpha\sigma_T \left(\frac{m}{6\pi T}\right)^{1/2} Z^2 n_i n_e \bar{g}_{ff} e^{-2\pi\nu/T} \\ &= 5.4 \times 10^{-40}\, T^{-1/2} Z^2 n_e n_i \bar{g}_{ff} e^{-h\nu/kT}, \end{aligned} \tag{12.22}$$

where the last line assumes SI units.

The total power emitted per unit volume is

$$\begin{aligned} \varepsilon_{ff} &= 4\pi \int_0^{\infty} j_\nu d\nu, \\ &= 8\pi\alpha\sigma_T \left(\frac{m}{6\pi T}\right)^{1/2} \int_0^{\infty} \bar{g}_{ff} e^{-2\pi\nu/T} d\nu \\ &= 4\alpha\sigma_T \left(\frac{mT}{6\pi}\right)^{1/2} Z^2 n_i n_e \bar{g}_B && (12.23) \\ &= 1.4 \times 10^{-28} T^{1/2} Z^2 n_e n_i \bar{g}_B(T), && (12.24) \end{aligned}$$

where $\bar{g}_B$ is $\bar{g}_{ff}$ averaged over frequency. It is typically in the range 1.1 to 1.5, with values near 1.2 being typical.

12.5 Free-free absorption

Radiation can also be absorbed by an electron moving in the electric field of an ion, with a corresponding increase in the energy of the electron. This is called free-free absorption. For thermal bremsstrahlung, the emission and absorption coefficients are related by Kirchhoffs law (11.2). Therefore the absorption coefficient is,

$$\begin{aligned} \alpha_{ff} &= \frac{j_\nu}{B_\nu} = \left(\frac{e^{\omega/T} - 1}{4\pi\nu^3}\right) 2\alpha\sigma_T \left(\frac{m}{6\pi T}\right)^{1/2} Z^2 n_i n_e \bar{g}_{ff} e^{-\omega/T}, \\ &= \frac{\alpha\sigma_T}{2\pi}\left(\frac{m}{6\pi T}\right)^{1/2} Z^2 n_i n_e \bar{g}_{ff} \nu^{-3}(1 - e^{-h\nu/kT}), \end{aligned} \tag{12.25}$$

In the Rayleigh-Jeans (low frequency) limit, $\alpha_{ff} \propto \nu^{-2}$, so free-free absorption cuts of the spectrum at low frequencies.