PHYS 350 – Problem Set #9 Solutions

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#1 Unloaded Loaded

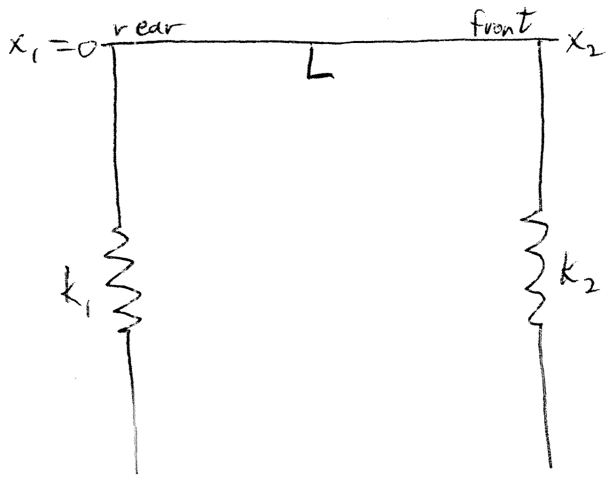

Fig. 1

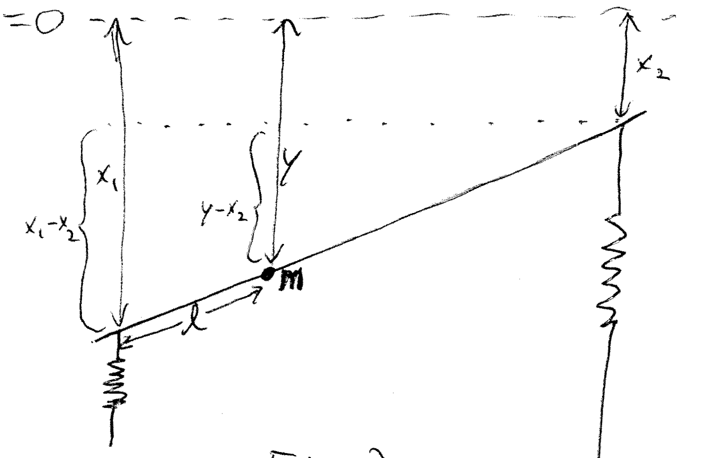

Fig. 2

↓ +ve direction

From geometry of Fig. 2

$$\frac{y-x_2}{L-\ell} = \frac{x_1-x_2}{L} \Rightarrow \boxed{\begin{aligned} y &= (1-\nu)x_1 + \nu x_2 \\ \nu &\equiv \frac{\ell}{L}\end{aligned}}$$

Potential Energy

$$V(x_1,x_2) = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 - mgy$$

$$= \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 - mg[(1-\nu)x_1 + \nu x_2]$$

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Equilibrium point (x_1°, x_2°) occurs when $\nabla V(x_1^\circ,x_2^\circ) = 0$

$$\left.\frac{\partial V}{\partial x_1}\right|_{\vec{x}^\circ} = 0 \Rightarrow k_1x_1^\circ - mg(1-\nu) = 0 \Rightarrow x_1^\circ = \frac{mg}{k_1}(1-\nu)$$

$$\left.\frac{\partial V}{\partial x_2}\right|_{\vec{x}^\circ} = 0 \Rightarrow k_2x_2^\circ - mg\nu = 0 \Rightarrow x_2^\circ = \frac{mg}{k_2}\nu$$

Expand V about (x_1°, x_2°) in Taylor Series

$$V(x_1,x_2) = V(x_1^\circ,x_2^\circ) + \sum_i \left.\frac{\partial V}{\partial x_i}\right|_{\vec{x}^\circ}(x_i-x_i^\circ) + \frac{1}{2}\sum_{i,j}\left.\frac{\partial^2 V}{\partial x_i\partial x_j}\right|_{\vec{x}^\circ}(x_i-x_i^\circ)(x_j-x_j^\circ)$$

$V(x_1^\circ, x_2^\circ) = \text{const}$ is arbitrary $\Rightarrow V(x_1^\circ, x_2^\circ) = 0$

$\left.\dfrac{\partial V}{\partial x_i}\right|_{\vec{x}^\circ} = 0$ by definition of $\vec{x}^\circ$

$$\left.\frac{\partial V}{\partial x_i \partial x_j}\right|_{\vec{x}^\circ} = k_i \delta_{ij}$$

$$\Rightarrow V(x_1, x_2) = \frac{1}{2} k_1 (x_1 - x_1^\circ)^2 + \frac{1}{2} k_2 (x_2 - x_2^\circ)^2$$

Define $\eta_i \equiv x_i - x_i^\circ$

$$\Rightarrow \boxed{V(\eta_1, \eta_2) = \vec{\eta} \cdot (\hat{V} \vec{\eta})}$$

where

$$\boxed{\hat{V} = \frac{1}{2} \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}}$$

Kinetic Energy

$$T = \frac{1}{2} m \dot{y}^2 = \frac{1}{2} m \left[(1-\nu) \dot{x}_1 + \nu \dot{x}_2 \right]^2$$

$$= \frac{1}{2} m \left[(1-\nu)^2 \dot{x}_1^2 + 2(1-\nu)\nu \dot{x}_1 \dot{x}_2 + \nu^2 \dot{x}_2^2 \right]$$

and, since $\dot{\eta}_i = \dot{x}_i$ we can write this as

$$\boxed{T = \dot{\vec{\eta}} \cdot (\hat{T} \dot{\vec{\eta}}), \quad \hat{T} = \frac{1}{2} m \begin{bmatrix} (1-\nu)^2 & \nu(1-\nu) \\ \nu(1-\nu) & \nu^2 \end{bmatrix}}$$

The normal mode frequencies are ω such that

$$|\hat{V} - \omega^2 \hat{T}| = 0$$

$$\Rightarrow \begin{vmatrix} k_1 - m(1-\nu)^2\omega^2 & -m\nu(1-\nu)\omega^2 \\ -m\nu(1-\nu)\omega^2 & k_2 - m\nu^2\omega^2 \end{vmatrix} = 0$$

(where $|\hat{A}| = \det(A)$). Solving for ω^2 gives

$$\boxed{\omega^2 = \frac{k_1 k_2}{m\left(\nu^2 k_1 + (1-\nu)^2 k_2\right)}}$$

The normal mode $\vec{a} = a_1\eta_1 + a_2\eta_2$ associated with this ω is

$$(\hat{V} - \omega^2\hat{T})\vec{a} = 0$$

$$\Rightarrow \begin{pmatrix} k_1 - m(1-\nu)^2\omega^2 & -m\nu(1-\nu)\omega^2 \\ -m\nu(1-\nu)\omega^2 & k_2 - m\nu^2\omega^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

Using the above expression for ω^2 and solving for a_1:

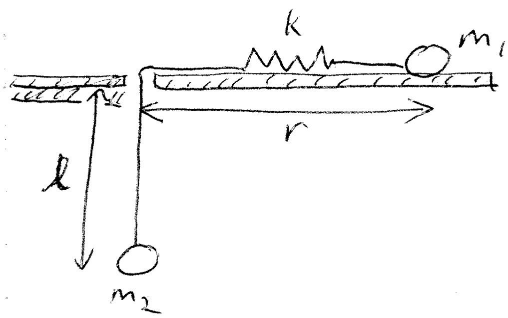
$$\boxed{a_1 = \frac{1-\nu}{\nu}\frac{k_2}{k_1}a_2}$$

Interpretation: η_1 describes motion of left (rear) end of car. The coefficient of $\eta_1(t)$ describes how much the left end moves in this normal mode of oscillation. Hence, to maximize rear-end oscillations we want to maximize a_1 w.r.t. a_2

$$\frac{a_1}{a_2} \gg 1 \Rightarrow \boxed{\frac{1-\nu}{\nu}\frac{k_2}{k_1} \gg 1} \Rightarrow \begin{array}{ll} \nu = \frac{\ell}{L} & \text{very small} \\ k_1 & \text{very small} \\ k_2 & \text{very large} \end{array}$$

i.e. to maximize rear-end oscillations, put all the mass in the back, put a loose spring in the back and a stiff spring up front.

#2 Side View



Top View

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Kinetic Energy

$$T = \frac{1}{2} m_1 \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\theta}^2 + \frac{1}{2} m_2 \dot{l}^2$$

Potential Energy

$$V = -m_2 g l + \frac{1}{2} k (r + l - b)^2$$

Lagrangian

$$\boxed{L = T - V = \frac{1}{2} m_1 (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} m_2 \dot{l}^2 + m_2 g l - \frac{1}{2} k (r + l - b)^2}$$

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Equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{d}{dt}(m_1 r^2 \dot{\theta}) = 0$$

$$\Rightarrow m_1 r^2 \dot{\theta} = p_\theta = \text{const.}$$

$$\Rightarrow \boxed{\dot{\theta} = \frac{p_\theta}{m_1 r^2}} \qquad (2.1)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

$$\Rightarrow m_1 \ddot{r} - m_1 r \dot{\theta}^2 + k(r + l - b) = 0$$

using (2.1)

$$m_1 \ddot{r} - m_1 r \dot{\theta}^2 + k(r+l-b) = 0$$

$$\Rightarrow \boxed{m_1 \ddot{r} - \frac{p_\theta^2}{m_1 r^3} + k(r+l-b) = 0} \qquad (2.2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{l}} - \frac{\partial L}{\partial l} = 0$$

$$\Rightarrow \boxed{m_2 \ddot{l} - m_2 g + k(r+l-b) = 0} \qquad (2.3)$$

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Steady State $(r_0, l_0) \Rightarrow \ddot{r} \equiv \ddot{l} \equiv 0$

$$\Rightarrow \begin{cases} k(r_0+l_0-b) = \dfrac{p_\theta^2}{m_1 r_0^3} & (2.4) \\ k(r_0+l_0-b) = m_2 g & (2.5) \end{cases}$$

$$\Rightarrow \boxed{\begin{array}{l} r_0^3 = \dfrac{p_\theta^2}{m_1 m_2 g} \Rightarrow r_0 = \dfrac{m_2 g}{m_1 \omega^2} \\ l_0 = b + \dfrac{m_2 g}{k} - \underbrace{\left(\dfrac{p_\theta^2}{m_1 m_2 g}\right)^{\frac{1}{3}}}_{r_0} \end{array}} \qquad \omega \equiv \dot{\theta}$$

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Expand about r_0, l_0 (p_θ remains constant of motion)

$$r = r_0 + \rho \qquad\qquad l = l_0 + \nu$$

and work to first order (in Equ. of motion!) in ρ and ν.

Eq. (2.2) becomes

$$m_1 \ddot{\rho} - \frac{p_\theta^2}{m_1 (r_0+\rho)^3} + k(\rho+\nu) + k(r_0+l_0-b) = 0$$

expanding second term to 1st order in ρ and using (2.4)

$$m_1 \ddot{\rho} - \frac{p_\theta^2}{m_1 r_0^3} + \frac{3 p_\theta^2}{m_1 r_0^4}\rho + k(\rho + \nu) + \frac{p_\theta^2}{m_1 r_0^3} = 0$$

$$\Rightarrow \boxed{m_1 \ddot{\rho} + \left(\frac{3 p_\theta^2}{m_1 r_0^4} + k\right)\rho + k\nu = 0}$$

Eq. (2.3) becomes

$$m_2 \ddot{\nu} - m_2 g + k(\rho + \nu) + k(r_0 + l_0 - b) = 0$$

using (2.5) we get

$$\boxed{m_2 \ddot{\nu} + k\nu + k\rho = 0}$$

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Compare to epicycle equations derived in class.

Using $p_\theta \simeq m_1 r_0^2 \omega + \mathcal{O}(\rho)$

$$m_1 \ddot{\rho} + (3 m_1 \omega^2 + k)\rho + k\nu = 0$$

$$m_2 \ddot{\nu} + k\rho + k\nu = 0$$