Astronomy 304 *Instructor: Jaymie Matthews*

PROBLEM SET 1 – *Exoplanets*
SOLUTIONS

Distributed: 8 February 2010
Due: Thursday, 4 March 2010 at the start of class

Star and planet formation

1. Stars are born when fragments of an interstellar nebula (molecular cloud) collapse due to a gravitational instability known as the *Jeans criterion*. This criterion can be expressed as a mass, so that the *Jeans mass* M_{Jeans} is given by

$$M_{Jeans} \text{ (in } M_{Sun}) \sim 23\, T^{3/2}\, n^{-1/2}$$

where T is cloud temperature (in K) and n is number density (in particles per cm^3).

(a) Derive this expression, assuming the cloud has a uniform temperature and density and that it is composed entirely of hydrogen. [12 marks]

The Virial Theorem for a system in gravitational equilibrium is that the total energy of the system is equal to one half of the gravitational potential energy of the system: $E_{tot} = \frac{1}{2}E_{grav}$

2 marks

The total energy is the sum of the kinetic energy E_{kin} *and the potential energy, so* $E_{kin} + E_{grav} = \frac{1}{2}E_{grav}$ *and therefore* $E_{kin} + \frac{1}{2}E_{grav} = 0$ *or* $2E_{kin} + E_{grav} = 0$

In the case of a gas cloud that is not expanding, contracting or convecting, the kinetic energy is represented by the thermal energy of the particles in it. (I'll neglect rotational energy which could alter the result slightly.)

The average kinetic energy of a particle in a gas of uniform temperature T is (3/2)kT where T is in Kelvin. If we assume the cloud is made entirely of hydrogen, then there is only one type of particle and the total kinetic energy of the nebula is $E_{kin} = N(3/2)kT$ *where N is the total # of H molecules.*

2 marks

The gravitational potential energy of a spherical distribution of matter of mass M and radius R is $E_{grav} = -(3/5)GM^2/R$

2 marks

So $2\,[\,N(3/2)kT\,] + [\,-(3/5)GM^2/R\,] = 0$

We want to express the equilibrium as a mass, so we must re-express other terms as mass.

The total number of particles N = (total mass) / (mass per particle)
$N = M / m_H$ where m_H is the mass of an H_2 molecule.

1 mark

The radius of the nebula $R = (3M / 4\pi\rho)^{1/3}$
and the density $\rho = m_H n$
where n is the number density of particles.

1 mark

$$2 [(M / m_H) (3/2)kT] + [-(3/5)GM^2 / (3M / 4\pi\, m_H n)^{1/3}] = 0$$

Isolating M in this equation …

1 mark

$$M = 4.6 \times 10^{34}\, T^{3/2}\, n^{-1/2} \text{ kg}$$

where you must be careful to use the mass of a hydrogen molecule (twice the mass of an H atom).

The number density is in particles per cubic metre so this must be converted to units of particles per cubic centimetre to match the equation I gave, and you must also convert to mass in units of solar masses ($M_{Sun} \sim 2 \times 10^{30}$ kg)

2 marks

1 mark

The final result is: $M_{Jeans} \sim 23\, T^{3/2}\, n^{-1/2}$ kg

(b) What is the Jeans mass for a cloud of temperature T = 30 K and number density $n = 10^{11}$ m^3. [3 marks]

The answer is about 12 solar masses. (Remember: You must convert number density into particles per cubic centimetre to apply the equation you derived in part (a).

2. Estimate how long it will take for a cloud fragment to collapse into a star, as a function of the initial number density of the nebula. [5 marks]

For most of the collapse, the individual molecules are falling close to freely under gravity towards the centre of mass, so we can use the free-fall timescale as an estimate. This is the same as half the orbital period P of a particle around a mass M with an orbital major axis 2a = R. By Newton's form of Kepler's 3rd Law: $P^2 = (4\pi^2/G(M_{tot}))\, a^3$ so $t_{ff} \sim ((\pi^2/8GM)R^3)^{1/2}$

Note that mass density $\rho = M/(4\pi R^3/3)$ so $\underline{t_{ff} \sim (3\pi/32G)^{1/2}\rho^{-1/2}}$
Voila! (There are other approaches I'll accept.)

5 marks

3. Before the star becomes a star, it is not radiating thermonuclear energy, but converting gravitational potential into thermal energy. What is the average luminosity (in units of L_{Sun}) of the protostellar cloud as it collapses? *[<u>8 marks</u>]*

There are various ways to approach this approximation. One is to calculate the loss of gravitational potential energy ($E_{grav} = -(3/5)GM^2/R$) from the initial size of the cloud to the final radius of the star.

$\Delta E_{grav} = -(3/5)GM^2(1/R_{cloud} - 1/R_{star})$

The initial size is so large that even if you calculate the Jeans length for the cloud, the initial potential energy will be close to zero ($R_{cloud} \sim \infty$). The final radius of the main sequence star will range somewhere between 0.1 and about 10 R_{Sun}.

The average luminosity $L \sim \Delta E_{grav} / t_{ff} \sim (3/5)GM^2/R_{star} / (3/32G)^{1/2}\rho^{-1/2}$

$L \sim 6.4\, G^{3/2}\, \rho^{1/2}\, M^2 / R_{star}$

For a typical density of the interstellar medium, a mass of 1 solar mass and a final radius of 1 solar radius, the average luminosity from this relation turns out to be about 10^{27} W ~ 3 L_{Sun}. (Keep in mind that most of this is being radiated in the infrared and far infrared for most of the protostellar collapse.)

<u>*The habitable zone*</u>

If a planet is to harbor LAWKI (Life As We Know It) then it must be able to have liquid water at or near its surface. Any planet whose orbit relative to its star allows the presence of liquid water is said to be in the "*<u>habitable zone</u>*".

4. Estimate the equilibrium temperature of a planet's surface as a function of its distance from its parent star, and any other relevant parameters of the star and the planet (e.g., star's luminosity, planet's radius and albedo). Be careful to point out any approximations or assumptions and justify their validity. *[<u>12 marks</u>]*

The flux received by a planet from its parent star is $F_{received} = L_{star} / 4\pi d^2$ where L_{star} is the luminosity of the star (in W) and d is the distance between the planet and star (in m). The total energy intercepted by the planet per second is received flux times area. But the area of the planet that is intercepting flux is its projected circular area $A = \pi R^2$ (where R is the radius of the planet), <u>not</u> half the surface area of a sphere.

4 marks

So the intercepted energy per time is $L_{intercepted} = (L_{star} / 4\pi d^2)(\pi R^2)$

Let's assume the planet radiates as a blackbody, so its radiated surface flux is $F_{radiated} = \sigma T^4$ where T is the planet's surface temperature (in K).

2 marks

The total radiated energy per time $L_{radiated} = F_{radiated} \times$ surface area = $\sigma T^4 \, 4\pi R^2$. (If the planet is rotating, it will radiate evenly on both hemispheres. If not, and there is no atmosphere, you could divide the radiating surface area by 2.)

2 marks

2 marks

The planet's surface will not absorb <u>all</u> the light it intercepts, but will reflect a fraction A, where A is the albedo, so $L_{absorbed} = (1 - A)\, L_{intercepted}$

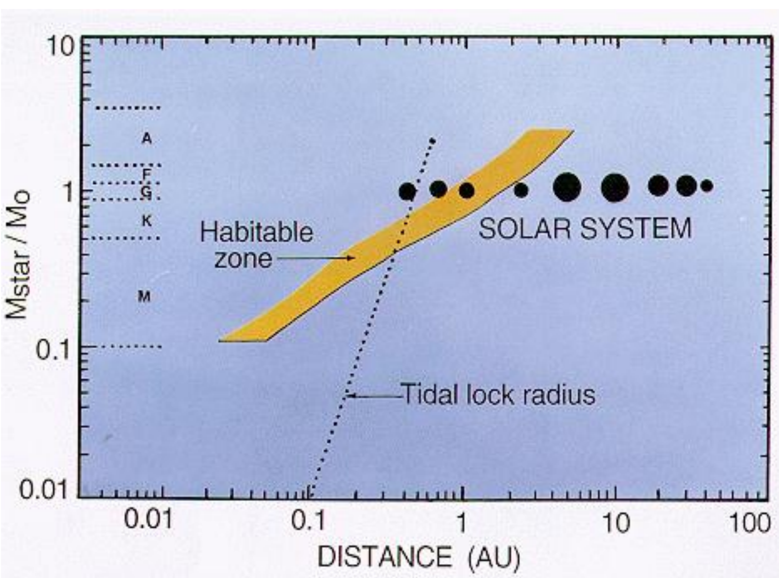
$$L_{absorbed} = (1 - A)(L_{star} / 4\pi d^2)(\pi R^2)$$

$$L_{absorbed} = L_{radiated}$$

$$(1 - A)(L_{star} / 4\pi d^2)(\pi R^2) = \sigma T^4 \, 4\pi R^2$$

Therefore, $T = [\, (1 - A)\, L_{star} /(16\pi\sigma)\,]^{1/4}\, d^{-1/2}$

5. Generate your own plot of the boundaries of the habitable zone as a function of spectral type for main sequence stars. Is there a limit (or are there limits) on stellar spectral type beyond which we would not expect to find simple or complex life even on a planet orbiting in its habitable zone? Why or why not? *[<u>13 marks</u>]*



As I discussed in class, the main sequence lifetime of stars shortens dramatically with increasing mass. By the time you reach spectral type A5, the lifetime of the star is less than 0.5 Gyr (half a billion years) which is the time between the appearance of oceans on Earth and the development of the first simple life forms. So planets orbiting stars more massive than mid-A-type probably won't have habitable conditions long enough for complex life to develop. (If the Earth's case is representative.)

TOTAL = 50 marks