Einstein's field equation, exterior and interior Schwarzschild solution: A general introduction

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We derive the gravitational field of a homogeneous spherically symmetric body ("star") in Newton's and in Einstein's gravitational theory, respectively. On our way to these results, we formulate Newton's theory in a quasi-field theoretical form, underline its incompatibility with special relativity theory, and point out how one arrives at Einstein's field equation. The gravitational field of the "Einsteinian" star consists of the *exterior* and the *interior* Schwarzschild solution which are joint together at the surface of the star. Their derivation and interpretation will be discussed.

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1.1 Newton's gravitational theory in quasi-field theoretical form

Gravity exists in all bodies universally and is proportional to the quantity of matter in each [. . .] If two globes gravitate towards each other, and their matter is homogeneous on all sides in regions that are equally distant from their centers, then the weight of either globe towards the other will be inversely as the square of the distance between the centers.
Isaac Newton (1687)

The gravitational force of a point–like mass m_2 on a similar one of mass m_1 is given by Newton's attraction law,

$$\boldsymbol{F}_{2\rightarrow 1} = -G\,\frac{m_1\,m_2}{|\boldsymbol{r}|^2}\,\frac{\boldsymbol{r}}{|\boldsymbol{r}|}\,, \tag{1.1}$$

where G is Newton's gravitational constant, see [8],

$$G \overset{\text{SI}}{=} 6.67559(27)\times 10^{-11}\,\frac{(\text{m/s})^4}{N}\,.$$

The vector $\boldsymbol{r} := \boldsymbol{r}_1 - \boldsymbol{r}_2$ points from m_2 to m_1, see figure 1.1. According to *actio* = *reactio* (Newton's 3rd law), we have $\boldsymbol{F}_{2\rightarrow 1} = -\boldsymbol{F}_{1\rightarrow 2}$. Thus a complete symmetry exists of the gravitational interaction of the two masses onto each other.

Let us now distinguish the mass m_2 as field–generating active gravitational mass and m_1 as (point–like) passive test–mass. Accordingly, we introduce a hypothetical *gravitational field* as describing the force per unit mass ($m_2 \hookrightarrow M\,,\ m_1 \hookrightarrow m$):

$$\boldsymbol{f} := \frac{\boldsymbol{F}}{m} = -\frac{G}{|\boldsymbol{r}^2|}\,\frac{\boldsymbol{r}}{|\boldsymbol{r}|}\,. \tag{1.2}$$

With this definition, the force acting on the test–mass m is equal to *field strength* × *gravitational charge* (mass) or $\boldsymbol{F}_{M\rightarrow m} = m\,\boldsymbol{f}$, in analogy to electrodynamics. The active gravitational mass M is thought to emanate a gravitational field which is always directed to the center of M and has the same magnitude on every sphere with M as center, see figure 1.2.

Let us now investigate the properties of the gravitational field (1.2). Obviously, there exists a potential

$$\phi = -G\,\frac{M}{|\boldsymbol{r}|}\,, \qquad \boldsymbol{f} = -\text{grad}\,\phi\,. \tag{1.3}$$

Accordingly, the gravitational field is curl-free: $\text{curl}\,\boldsymbol{f} = 0$.

Figure 1.1. Two mass points m_1 and m_2 in 3-dimensional space, Cartesian coordinates x, y, z.

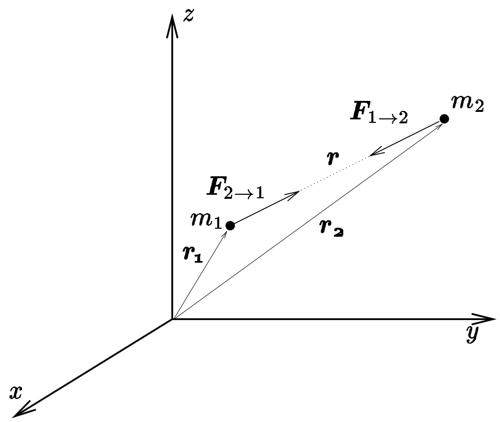
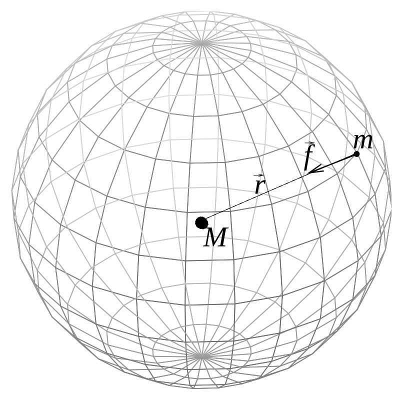


Figure 1.2. The "source" M attracts the test mass m.



By assumption it is clear that the source of the gravitational field is the mass M. We find, indeed,

$$\operatorname{div} \boldsymbol{f} = -4\pi G M \, \delta^3(\boldsymbol{r}) \,, \tag{1.4}$$

where $\delta^3(\boldsymbol{r})$ is the 3-dimensional (3D) delta function. By means of the *Laplace operator* $\Delta :=$ div grad, we infer for the gravitational potential

$$\Delta \, \phi = 4\pi \, G \, M \, \delta^3(\boldsymbol{r}) \,. \tag{1.5}$$

The term $M \, \delta^3(\boldsymbol{r})$ may be viewed as the mass density of a point mass. Equation (1.5) is a 2nd order linear partial differential equation for ϕ.Thus the gravitational potential generated by several point masses is simply the linear superposition of the respective single potentials. Hence we can generalize the *Poisson equation* (1.5) straightforwardly to a continuous matter distribution $\rho(\boldsymbol{r})$:

$$\Delta \, \phi = 4\pi \, G \, \rho \,. \tag{1.6}$$

This equation interrelates the source ρ of the gravitational field with the gravitational potential ϕ and thus completes the quasi-field theoretical description of Newton's gravitational theory.

We speak here of *quasi*–field theoretical because the field ϕ as such represents a convenient concept. However, it has no *dynamical* properties, no genuine degrees of freedom. The Newtonian gravitational theory is a *action at a distance* theory. When we remove the source, the field vanishes instantaneously. Newton himself was very unhappy about this consequence. Therefore he emphasized the preliminary and purely descriptive character of his theory. But before we liberate the gravitational field from this constraint by equipping it with its own degrees of freedom within the framework of general relativity theory, we turn to some properties of the Newtonian theory.

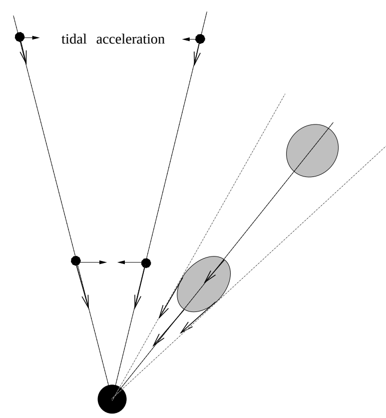
A very peculiar fact characteristic to the gravitational field is that the acceleration of a freely falling test-body does not depend on the mass of this body but only on its position within the gravitational field. This comes about because of the equality (in suitable units) of the gravitational and the inertial mass:

$$\overset{\text{inertial}}{m}\;\ddot{\boldsymbol{r}} = \boldsymbol{F} = \overset{\text{grav}}{m}\;\boldsymbol{f}\,. \tag{1.7}$$

This equality has been well tested since Galileo's time by means of pendulum and other experiments with an ever increasing accuracy, see Will [21].

In order to allow for a more detailed description of the structure of a gravitational field, we introduce the concept of *tidal force*. This can be best illustrated by means of figure 1.3. In a spherically symmetric gravitational field, for example, two test-masses will fall radially towards the center and thereby get closer and closer. Similarly, a spherical drop of water is deformed to an ellipsoidal shape because the gravitational force at its bottom is bigger than at its top, which has a greater distance to the source. If the distance between two freely falling test masses is relatively

Figure 1.3. Tidal forces in a spherically symmetric gravitational field



small, we can derive an explicit expression for their relative acceleration by means of a Taylor expansion. Consider two mass points with position vectors $\boldsymbol{r}$ and $\boldsymbol{r}+\delta\boldsymbol{r}$, with $|\delta\boldsymbol{r}| \ll 1$. Then the relative acceleration reads

$$\Delta\boldsymbol{a} = [\boldsymbol{f}(\boldsymbol{r}+\delta\boldsymbol{r}) - \boldsymbol{f}(\boldsymbol{r})] = \delta\boldsymbol{r}\cdot\operatorname{Grad}\boldsymbol{f}(\boldsymbol{r})\,, \tag{1.8}$$

where Grad denotes the vector gradient. We may rewrite this according to (the sign is conventional, $\partial/\partial x^a =: \partial_\alpha,\ x^1 = x,\ x^2 = y,\ x^3 = z$)

$$K_{ab} := -\left(\operatorname{Grad}\boldsymbol{f}\right)_{ab} = -\partial_a f_b\,, \qquad a,b = 1,2,3\,.$$

We call K_{ab} the *tidal force* matrix. The vanishing curl of the gravitational field is equivalent to its symmetry, $K_{ab} = K_{ba}$. Furthermore, $K_{ab} = \partial_a\,\partial_b\,\phi$. Thus, the Poisson equation becomes,

$$\sum_{a=1}^{3} K_{aa} = \operatorname{trace} K = 4\pi\, G\,\rho\,. \tag{1.9}$$

Accordingly, in vacuum K_{ab} is trace-free.

Let us now investigate the gravitational potential of a homogeneous star with constant mass density $\rho_\odot$ and total mass $M_\odot = (4/3)\,\pi\,R_\odot^3\,\rho_\odot$. For our sun, the radius is $R_\odot = 6.9598\times10^8\,\mathrm{m}$ and the total mass is $M = 1.989\times10^{30}\,\mathrm{kg}$.

Outside the sun (in the idealized picture we are using here), we have vacuum. Accordingly, $\rho(\boldsymbol{r}) = 0$ for $|\boldsymbol{r}| > R_\odot$. Then the Poisson equation reduces to the *Laplace equation*

$$\Delta\,\phi = 0\,, \qquad \text{for } r > R_\odot\,. \tag{1.10}$$

In 3D polar coordinates, the r-dependent part of the Laplacian has the form $(1/r^2)\,\partial_r\,(r^2\,\partial_r)$. Thus (1.10) has the solution

$$\phi = \frac{\alpha}{r} + \beta\,, \tag{1.11}$$

where α and β are integration constants. Requiring that the potential tends to zero as r goes to infinity, we get $\beta = 0$. The integration constant α will be determined from the requirement that the force should change smoothly as we cross the star's surface, i.e. the interior- and exterior potential and their first derivatives have to be matched continuously at $r = R_\odot$.

Inside the star we have to solve

$$\Delta\,\phi = 4\pi\,G\,\rho_\odot\,, \qquad \text{for } r \le R_\odot\,. \tag{1.12}$$

We find

$$\phi = \frac{2}{3}\,\pi G \rho_\odot\,r^2 + \frac{C_1}{r} + C_2\,, \tag{1.13}$$

with integration constants C_1 and C_2. We demand that the potential in the center $r = 0$ has a finite value, say ϕ_0. This requires C_1=0. Thus

$$\phi = \frac{2}{3}\pi\,G\,\rho_\odot\,r^2 + \phi_0 = \frac{G\,M(r)}{2r} + \phi_0\,, \tag{1.14}$$

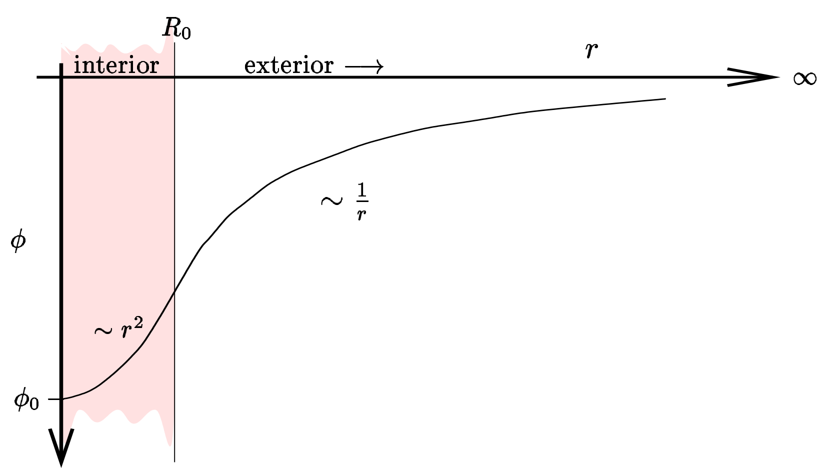
where we introduced the *mass function* $M(r) = (4/3)\,\pi r^3 \rho_\odot$ which measures the total mass inside a sphere of radius r.

Continuous matching of ϕ and its first derivatives at $r = R_\odot$ finally yields:

$$\phi(\boldsymbol{r}) = \begin{cases} -G\,\dfrac{M_\odot}{|\boldsymbol{r}|} & \text{for} \quad |\boldsymbol{r}| \ge R_\odot\,, \\ G\,\dfrac{M_\odot}{2R_\odot^3}\,|\boldsymbol{r}|^2 - \dfrac{3G\,M_\odot}{2R_\odot} & \text{for} \quad |\boldsymbol{r}| < R_\odot\,. \end{cases} \tag{1.15}$$

The slope of this curve indicates the magnitude of the gravitational force, the curvature (2nd

Figure 1.4. Newtonian potential of a homogeneous star.



derivative) the magnitude of the tidal force (or acceleration).

1.2 Special relativity and Newton's theory: A clash

Not only have we no direct intuition of the equality of two periods, but we have not even direct intuition of the simultaneity of two events occurring in two different places.

Henri Poincaré (1902)

Apparently, the space surrounding us has 3 dimensions. Together with the 1 dimensional time, it constitutes *4-dimensional* (4D) *spacetime.* Distinguished frames of reference are the inertial frames. They are understood as infinitely extended frames in which *forcefree* particles are at rest or move uniformly along straight lines in the sense of Euclidean geometry. In them, we may introduce coordinates

$$x^0 = ct\,,\ x^1 = x\,,\ x^2 = y\,,\ x^3 = z\,,\ \text{or } x^\mu\,, \qquad \text{with } \mu = 0,1,2,3\,. \tag{1.16}$$

As a rule, all Greek indices shall run form 0 to 3. In empty space with respect to an *inertial* frame of reference, there is no distinction of different points in it and no preferred direction. Likewise, there is no preferred instant of time.

With this homogeneous and spatially isotropic spacetime in mind, we state the *special relativity principle:* The laws of physics are the same in all inertial frames.

A prototypical law of nature to be stated in this context is the *principle of the constancy of the speed of light:* Light signals in vacuum are propagated rectilinearly, with same speed c at all times, in all directions, in all inertial frames, independent of the motion of their sources.

By means of these two principles, we can deduce the *Poincaré* (or inhomogeneous Lorentz) transformations which encompass 4 spacetime translations, 3 spatial rotations, and 3 *Lorentz boosts*, i.e., velocity transformations. The "essence" of this transformation can be expressed also in a somewhat different manner.

We define a tensor $\mathbf{T}$ of covariant rank k and contravariant rank l, respectively by means of its behavior under coordinate transformations,

$$T^{\mu_1{}'\cdots\mu_l{}'}{}_{\nu_1{}'\cdots\nu_k{}'} = P^{\mu_1'}_{\mu_1}\cdots P^{\mu_l'}_{\mu_l}\,P^{\nu_1}_{\nu_1'}\cdots P^{\nu_k}_{\nu_k'}\,T^{\mu_1\cdots\mu_l}{}_{\nu_1\cdots\nu_k}\,, \tag{1.17}$$

where we introduced the *Jacobian* matrix and its inverse according to

$$P^{\alpha'}_{\alpha} := \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\,,\quad P^{\alpha}_{\alpha'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\,,\quad P^{\alpha}_{\alpha'}\,P^{\alpha'}_{\beta} = \delta^{\alpha}_{\beta}\,. \tag{1.18}$$

The summation convention is assumed, i.e., summation is understood over repeated indices. The value of the components of tensors do change, but only in the specific linear and homogeneous manner indicated above. Equations of tensors remain form invariant or covariant, that is, the transformed equations look the same but with the unprimed indices replaced by primed ones. If one *contracts* co- and contravariant tensors in such a way that no free index is left, $v_i\,w^i$, e.g., one gets a scalar, which is *invariant* under transformations, that is, it does not change its value. The latter represents an observable quantity. The generic case of a covariant tensor of first rank is the partial derivative of a scalar function $\phi_{,\alpha} := \partial\phi/\partial x^\alpha$ and the typical contravariant tensor is the coordinate differential dx^α. Besides tensors, we need also spinors in special relativity, but they are not essential in gravitational theory.

We define the *Minkowski metric* according to

$$ds^2 := -c^2\,dt^2 + dx^2 + dy^2 + dz^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta\,, \tag{1.19}$$

where (in Cartesian coordinates)

$$g_{\alpha\beta} \stackrel{*}{=} \eta_{\alpha\beta} := \mathrm{diag}\,(-1,+1,+1,+1) = \eta^{\alpha\beta} \stackrel{*}{=} g^{\alpha\beta}\,. \tag{1.20}$$

The $g^{\alpha\beta}$ denote the inverse of the metric tensor. Under a Poincaré-transformation, the components of the Minkowski metric $\eta_{\alpha\beta}$ remain numerically invariant. This metric defines an invariant spatiotemporal distance between two spacetime points or events, as they are called. Spatial distance alone between two points can be different for different observers and the same applies to time intervals. This manifests itself in the well-known effects of time dilation and length contraction.

Now we are able to express the principle of special relativity in the following way: The equations of physics describing laws of nature shall transform covariantly under Poincaré-transformations.

How can we apply this to gravity? In Newtonian gravity, the potential obeys the Poisson equation $\Delta\,\phi = 4\pi G\,\rho$. The corresponding wave equation can be represented as

$$\Box\,\phi = \partial_\alpha\,\left(\eta^{\alpha\beta}\,\partial_\beta\,\phi\right) = -\frac{1}{c^2}\,\frac{\partial^2\,\phi}{\partial t^2} + \Delta\,\phi = 4\pi G\,\rho\,, \tag{1.21}$$

and thus is manifestly Poincaré invariant. Hence, the Poisson equation as such is *not* Poincaré invariant but only a limiting case of the wave equation for static situations.

The first idea for a Poincaré covariant equation for the gravitational potential would be the obvious generalization by admitting the gravitational potential ϕ and the source ρ to be time-dependent and interrelating both by means of a gravitational wave equation $\Box\,\phi = 4\pi G\,\rho$. But what is the source ρ now? In the case of a pressure-less fluid or a swarm of dust particles where all components move parallelly with the same velocity (and correspondingly have a common rest system), there can be found a Poincaré invariant meaning of mass density, but this is not possible in general. Moreover, we learn from special relativity that mass and energy are equivalent. Binding forces and therewith stress within matter are expected to contribute to its gravitating mass. Thus, in a relativistic theory of gravitation, we have to replace mass density by energy density. Next, we have to look for a Poincaré invariant quantity which contains the (mass-)energy density and will reduce to it in special cases.

And indeed, special relativity provides such a quantity. In electrodynamics, Minkowski found a symmetric second rank tensor $T^{\alpha\beta}_{\text{Max}}$ whose divergence yields the Lorentz force density $\partial_\alpha\,T^{\alpha\beta} = f^\beta$. For an electrically charged perfect fluid, characterized by mass-energy density ρ and pressure p, the equations of motion can be written in the form

$$\partial_\alpha\,\left(T^{\alpha\beta}_{\text{Max}} + T^{\alpha\beta}_{\text{Mat}}\right) = 0\,, \tag{1.22}$$

where we introduced the *energy momentum tensor* of the perfect fluid

$$T^{\alpha\beta}_{\text{Mat}} = \left(\rho + \frac{p}{c^2}\right)\,u^\alpha\,u^\beta + p\,g^{\alpha\beta}\,. \tag{1.23}$$

The vector $u^\alpha = dx^\alpha/d\tau = \gamma(v)\,(c, \boldsymbol{v})$ is the four-velocity of the fluid elements (and $\boldsymbol{v}$ the three-velocity with respect to the chosen frame of reference. The *Lorentz factor* γ is given by $\gamma(v) := (1 - v^2/c^2)^{-1/2})$. The components of the energy momentum tensor are not invariant, of course. In the rest frame of the fluid, the observer sees a fluid at rest with a certain mass-energy distribution and an isotropic pressure p: $T^{\alpha\beta} \stackrel{*}{=} \text{diag}\,(\rho c^2, p, p, p)$. However, with respect to a moving frame, there is a moving energy distribution which results in an energy flux density. Moreover, isotropic pressure transforms into anisotropic stress etc. In general, we arrive at the following structure (momentum flux density and stress are equivalent notions, $i, j = 1, 2, 3$),

$$T_{\mu\nu} = \left(\begin{array}{c|c} T_{00} & T_{0i} \\ \hline T_{i0} & T_{ij} \end{array}\right) = \left(\begin{array}{c|c} \text{energy density} & \text{momentum density} \\ \hline \text{energy flux density} & \text{momentum flux density} \end{array}\right)\,. \tag{1.24}$$

Now we can construct a scalar invariant encompassing the mass-energy density in the following way

$$T := T_\alpha{}^\alpha = g_{\alpha\beta}\,T^{\alpha\beta} = -\rho c^2 + 3p\,. \tag{1.25}$$

For "non relativistic matter", we find $\rho \ll 3p/c^2$. Thus, indeed, $T \approx \rho c^2$. The Poincaré invariant field equation

$$\Box\,\phi = \kappa\,T \tag{1.26}$$

then yields the Newtonian Poisson equation in an appropriate limiting case and for a proper chosen coupling constant κ.

At first sight, this defines a viable gravitational theory. However, it turns out that this theory runs into serious conflicts with observations. A scalar gravitational theory does not allow for the deflection of light in the gravitational fields because a scalar field cannot be coupled reasonably to the electromagnetic field, since the electromagnetic energy-momentum tensor is traceless. Light deflection today is experimentally confirmed beyond doubt. Thus, we have to look for different possibilities in order to interrelate electromagnetic energy-momentum and the gravitational potential. To this end we will now turn to the gravitational field.

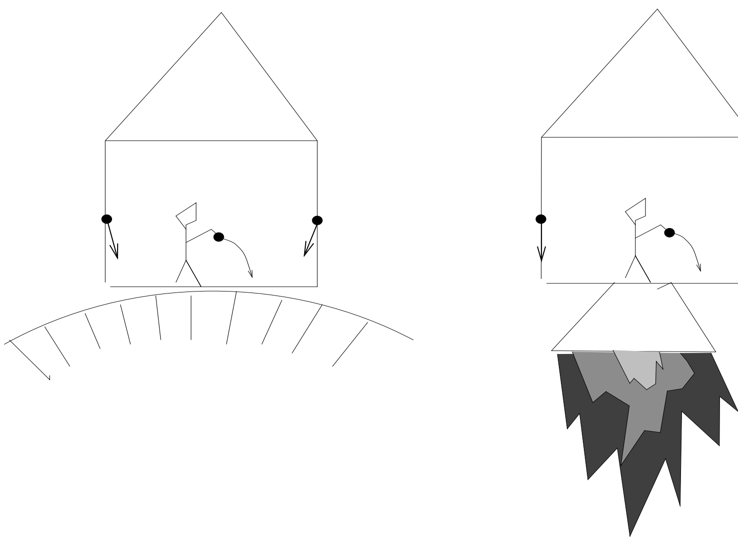
1.3 Accelerated frames of reference, equivalence principle and Einstein's field equation

Die Relativitätstheorie bringt uns aber nicht nur den Zwang, Newtons Theorie zu modifizieren; sie schränkt auch zum Glück in weitgehendem Maße die Möglichkeiten hierfür ein.

Albert Einstein (1913)

An observer who measures the acceleration of a freely falling body within a sufficiently small laboratory, gets the same results whether his lab is at rest in a gravitational field or appropriately accelerated in gravity free space. Consequently, the quantity representing the inertial forces in the equation of motion should be similar to the quantity representing the gravitational forces. In an

Figure 1.5. The local equaivalence of an accelerated frame of reference and a gravitational field. Note, if we compare the gravitational and the inertial forces acting on *two* point particles in each case, because of the tidal effect, we can distinguish the lab on earth and that in space. However, locally, *one* test particle moves in the same way in both labs.



inertial frame in Cartesian coordinates x^μ, a force-free test particle obeys the equation of motion

$$m\,\frac{d^2\,x^\mu}{d\tau^2} = 0\,. \tag{1.27}$$

Thus it moves in a straight line $x^\mu(\tau) = a^\mu + b^\mu\,\tau$ (a^μ, b^μ constant vectors). The space laboratory represents an accelerated frame of reference with coordinates $x^{\mu'}$. We apply a coordinate transformation $x^{\alpha'}(x^\mu)$ to (1.27) and find

$$m\,\frac{d^2 x^{\alpha'}}{d\tau^2} + m\,\Gamma^{\alpha'}{}_{\beta'\gamma'}\,\frac{dx^{\beta'}}{d\tau}\,\frac{dx^{\gamma'}}{d\tau} = 0\,, \tag{1.28}$$

where the *connection* components

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} \stackrel{*}{=} \frac{\partial x^{\alpha'}}{\partial x^\alpha}\,\frac{\partial^2 x^\alpha}{\partial x^{\beta'}\,\partial x^{\gamma'}} \tag{1.29}$$

represent the *inertial field.* For a rotating coordinate system, e.g., $\mathbf{\Gamma}$ encompasses the Coriolis force etc. So far $\Gamma^{\alpha'}{}_{\beta'\gamma'}$ is only an coordinate artifact and has no own degrees of freedom. We can always introduce a global coordinate system such that the $\Gamma^{\alpha'}{}_{\beta'\gamma'}$ vanish everywhere.

We can deduce an alternative representation of $\Gamma^{\alpha'}{}_{\beta'\gamma'}$ from the tensorial transformation behavior of the metric tensor (we suppress the dashes here):

$$\Gamma^\alpha{}_{\mu\nu} := \frac{1}{2}\,g^{\alpha\beta}\,\left(\partial_\nu\,g_{\beta\mu} + \partial_\mu\,g_{\beta\nu} - \partial_\beta\,g_{\mu\nu}\right)\,. \tag{1.30}$$

Thus, the connection components, also called *Christoffel symbols* in the case of a Riemannian space, can be expressed in terms of ten functions $g_{\alpha\beta} = g_{\beta\alpha}$ which tentatively serve as gravitational or inertial potential. In order to be able to choose a coordinate system such that $\Gamma^{\alpha'}{}_{\beta'\gamma'} = 0$ globally, the $\Gamma^{\alpha}{}_{\beta\gamma}$ have to fulfill a certain integrability condition, namely their "curl" has to vanish

$$0 = R^{\mu}{}_{\nu\alpha\beta} := \partial_{\alpha}\,\Gamma^{\mu}{}_{\nu\beta} - \partial_{\beta}\,\Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\sigma\alpha}\,\Gamma^{\sigma}{}_{\nu\beta} - \Gamma^{\mu}{}_{\sigma\beta}\,\Gamma^{\sigma}{}_{\nu\alpha}\,. \tag{1.31}$$

The quantity $R^{\alpha}{}_{\beta\mu\nu}$ is called the *Riemannian curvature* tensor. If $R^{\alpha}{}_{\beta\mu\nu} = 0$, we have a flat Minkowski space (possibly in curvilinear coordinates), whereas $R^{\alpha}{}_{\beta\mu\nu} \neq 0$ implies a *curved Riemannian spacetime*. In a Riemannian space, the curvature tensor fulfills certain *algebraic* identities which reduce its number of independent components to 20:

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}\,,\;\; R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}\,,\;\; R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0\,. \tag{1.32}$$

Let us now construct the field equation for gravity by trying to proceed along the same line as in other successful field theories, such as electrodynamics. The equations of motion with the abbreviation $\dot{(\;)} = d/d\tau$ read:

$$\begin{array}{lcccccc} \text{Maxwell:} & m\,\ddot{x}^{\alpha} & = & q\,\dot{x}^{\mu} & \times & F^{\alpha}{}_{\mu} \\ & & & \text{electric} & & \text{el.-mag.} \\ & & & \textit{current} & & \textit{field strength} \\ & & & \text{inertial} & & \text{inertial} \\ \text{Gravitation:} & m\,\ddot{x}^{\alpha} & = & -m\,\dot{x}^{\mu}\,\dot{x}^{\nu} & \times & \Gamma^{\alpha}{}_{\mu\nu} \end{array} \tag{1.33}$$

This fits quite nicely into our considerations of the last section. The current, which couples to the inertial field, is the quantity $m\,\dot{x}^{\mu}\,\dot{x}^{\nu}$ which corresponds to the energy-momentum tensor of dust $T^{\alpha\beta} = \rho\,\dot{x}^{\alpha}\,\dot{x}^{\beta}$. This coincides with the earlier suggestion that $T^{\alpha\beta}$ should be the source of gravity.

In electrodynamics, we have the four-potential $A_{\mu} = (\phi_{\text{elec}}, \boldsymbol{A})$, ϕ_{elec} is the 3D scalar electric potential, $\boldsymbol{A}$ the 3D vector potential. Furthermore, the electromagnetic field strength is denoted by $F_{\alpha\beta} = \partial_{\alpha}\,A_{\beta} - \partial_{\beta}\,A_{\alpha}$ and the current by J^{α}. With the *Lorenz gauge*, $\partial_{\mu}\,A^{\mu} = 0$, we find

$$\begin{array}{ccccc} \text{divergence of } \textit{field} & \sim & \text{d'Alembertian on } \textit{potential} & \sim & \textit{source current} \\ \partial_{\mu}\,F^{\mu\nu} & = & \Box\,A^{\nu} & = & J^{\mu}\,. \end{array} \tag{1.34}$$

However, it is not so simple in gravity. Gravitational radiation carries energy, and energy is, as we have argued above, itself a source of gravity. Thus, there is a self-interaction of the gravitational field which distinguishes it from the electromagnetic field. Consequently, gravity is described by a non-linear field equation of the following type

$$\text{"Div"}\,\boldsymbol{\Gamma} + \boldsymbol{\Gamma}^2 \quad \sim \quad \Box\,g_{\alpha\beta} + \text{nonlinear} \quad \sim \quad T_{\alpha\beta} \tag{1.35}$$

That the nonlinearity is only quadratic, will be a result of our subsequent considerations.

So much for the general outline. To fix an exact equation, we need some additional criteria. In particular, we have to say something about general covariance. We consider an accelerated frame of reference locally equivalent to one which is at rest in a gravitational field. Gravity is a relatively weak force, but it has an infinite range and is all-pervading. We will hardly find a gravity free spot in the universe. Hence, in general we find ourselves in a non-inertial frame, even if the deviation from an inertial system may be negligible on small scales. From this point of view, the fundamental laws of physics should be covariant not only under Poincaré transformations but under general coordinate transformations. There is not much change with respect to the algebra of tensors, but a very noticeable change comes about in tensor analysis: the partial derivative of a tensor will not transform like a tensor. This can be fixed by introducing the so-called *covariant derivative*:

$$\nabla_{\alpha}\,T^{\mu}{}_{\nu} = \partial_{\alpha}\,T^{\mu}{}_{\nu} + \Gamma^{\mu}{}_{\gamma\alpha}\,T^{\gamma}{}_{\nu} - \Gamma^{\gamma}{}_{\nu\alpha}\,T^{\mu}{}_{\gamma}\,. \tag{1.36}$$

By replacing the partial derivatives in the special relativistic formulae by covariant ones, we obtain general covariant equations. This "correspondence" principle mostly, but not always, yields physical reasonable generalizations of the special relativistic laws.

In Newton's theory, the mass density as source is linearly related to the tidal force. Can we also define tidal forces in general relativity?

The equation of motion (1.27) has a geometrical interpretation, too. The metrics provides for an invariant length of a curve $x^i(\tau)$ (τ is a parameter) connecting two spacetime points $\mathsf{A} = x^\mu(0)$ and $\mathsf{B} = x^\mu(\tau_0)$ by means of the line integral

$$l = \int_{x^i} ds = \int_0^{\tau_0} d\tau \, \sqrt{\dot{x}^\mu \, \dot{x}^\nu \, g_{\mu\nu}} \, . \tag{1.37}$$

This length l represents the proper time of an observer who moves along the path γ from A to B. The necessary and sufficient condition for γ to be a curve of extremal length is found to be (provided γ is parametrized by its arc length)

$$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta} \, \dot{x}^\alpha \, \dot{x}^\beta = 0 \, . \tag{1.38}$$

This is the *Euler-Lagrange* equation for the variational Problem $\delta \int ds = 0$, and it coincides with the equation of motion (1.27). In geometry, (1.38) is called *geodesic equation* and its solutions x^μ are *geodesics.* In flat space, the geodesics are straight lines, geodesics of a sphere are circuits, e.g.

Thus, freely falling particles move along geodesics of Riemannian spacetime. Now we can address the question of tidal accelerations between two freely falling particles. Let the vector v^μ be the vector describing the distance between two particles moving on infinitesimally adjacent geodesics. A simple calculation yields

$$\frac{D^2 \, v^\mu}{D\tau^2} = \dot{x}^\nu \, \dot{x}^\alpha v^\beta \, R^\mu{}_{\nu\alpha\beta} \, , \tag{1.39}$$

where $D/D\tau$ denotes the absolute derivative along the curve x^α. Eventually, the tidal acceleration is represented by the curvature tensor. In Newton's theory, the tidal force is linearly related to the tidal acceleration. The energy-momentum tensor, as the suspected source of gravity, is a symmetric 2nd rank tensor. Therefore it has 10 independent components.

It now remains the problem of how to interrelate the 2nd rank symmetric energy-momentum tensor to the 4th rank Riemannian curvature tensor. In analogy to the Newtonian case we would like this relation to be linear. It turns out that such a relation has to be of the form

$$\alpha \, R_{\mu\nu} + \beta \, R \, g_{\mu\nu} = T_{\mu\nu} \, . \tag{1.40}$$

with the *Ricci tensor*

$$R_{\alpha\beta} := R^\mu{}_{\alpha\mu\beta} \tag{1.41}$$

and the *curvature scalar*

$$R := R^\alpha{}_\alpha \, . \tag{1.42}$$

The constants α and β have to be fixed by additional conditions. The vanishing divergence of the energy-momentum together with the second Bianchi-identity (a kind of integrability condition)

$$\nabla_\lambda \, R^\alpha{}_{\beta\mu\nu} + \nabla_\nu \, R^\alpha{}_{\beta\lambda\mu} + \nabla_\mu \, R^\alpha{}_{\beta\nu\lambda} = 0 \tag{1.43}$$

leads to *Einstein's field equation*:

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu}}_{\text{Einstein tensor } G_{\mu\nu}} \quad + \quad \Lambda \, g_{\mu\nu} = \kappa \, T_{\mu\nu} \, . \tag{1.44}$$

The value $\kappa := \frac{8\pi G}{c^4}$ of *Einstein's gravitational constant* can be determined by transition to the Newtonian limit of general relativity. Moreover we added the cosmological term containing the *cosmological constant* Λ.

The energy-momentum tensor has 10 independent components whereas the Riemannian curvature tensor has 20 independent components. Hence, the energy-momentum tensor determines only a part of the curvature. Indeed, we have a decomposition

$$R_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta} + \frac{1}{2} \left(g_{\mu\alpha} \, L_{\beta\nu} - g_{\mu\beta} \, L_{\alpha\nu} - g_{\nu\alpha} \, L_{\beta\mu} + g_{\nu\beta} \, L_{\alpha\mu} \right) \, , \tag{1.45}$$

where

$$L_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{6} R \, g_{\alpha\beta} = L_{\beta\alpha} \tag{1.46}$$

(for recent work on the L-tensor, see [11]). This part of the curvature is algebraically linked to the matter distribution by means of Einstein's equation. Consequently, it vanishes in vacuum and there only remains the irreducible 4th rank piece $C_{\alpha\beta\gamma\delta}$, the conformal *Weyl curvature* with 10 independent components.

1.4 The exterior Schwarzschild solution

Es ist eine ganz wunderbare Sache, dass von einer so abstrakten Idee aus die Erklärung der Merkuranomalie so zwingend herauskommt.
Karl Schwarzschild (1915)

Just a few months after Einstein had published his new gravitational theory, the astronomer K. Schwarzschild found an exact solution to Einstein's field equation. The so-called Schwarzschild solution is amazingly simple, especially in view of the field equations which are very complicated. However, the Schwarzschild solution is not a degenerated case for over-simplified situations but physically most meaningful. It is this solution by means of which one can explain most general relativistic effects in the planetary system. The reason is that it describes the gravitational field outside of a spherically symmetric body — like the planets and the sun.

We start from the spherically symmetric metric:

$$ds^2 = -\mathrm{e}^{\lambda(r,t)}\, c^2\, dt^2 + \mathrm{e}^{\nu(r,t)}\, dr^2 + r^2\, d\Omega^2\,, \quad d\Omega := d\theta^2 + sin^2\, d\varphi^2\,. \tag{1.47}$$

One can now compute the Christoffel symbols, the Riemann tensor, and the Einstein tensor for this ansatz. This can be done by hand, of course. It is more convenient to use computer algebra, see the appendix. For vacuum and $\Lambda = 0$, it is relatively simple to find a solution to $G_{\alpha\beta} = \kappa\, T_{\alpha\beta} = 0$, namely

$$ds^2 = -\left(1 - \frac{2m}{r}\right) c^2\, dt^2 + \frac{1}{1 - \frac{2m}{r}}\, dr^2 + r^2\, d\Omega^2\,. \tag{1.48}$$

This is the *Schwarzschild metric* [19]. There is no time dependence although we did allow for that in the ansatz (1.47). The vacuum spacetime structure generated by any spherically symmetric body is *static*. This applies also for the exterior field of a radially oscillating body. This fact is known as *Birkhoff's theorem.*

The parameter $2m$ is an integration constant. Its interpretation can be obtained by means of transition to Newton's theory. It turns out that

$$r_{\mathrm{S}} := 2m = \frac{2GM}{c^2}\,, \tag{1.49}$$

where G is Newton's gravitational constant and M is the mass of gravitating body. At the *Schwarzschild radius* r_{S} the metric coefficients become singular. However, this is only a so-called *coordinate singularity* since the curvature tensor (and therewith physically meaningful quantities like the tidal forces) remains finite. We can also see this explicitly when we introduce suitable coordinates, like *isotropic coordinates.* Therefore we define a new radial coordinate $\bar{r}$ according to

$$r = \bar{r}\left(1 + \frac{m}{2\bar{r}}\right)^2\,. \tag{1.50}$$

Then, the Schwarzschild metric becomes

$$ds^2 = \left(\frac{1 - \frac{m}{2\bar{r}}}{1 + \frac{m}{2\bar{r}}}\right)^2 c^2\, dt^2 - \left(1 + \frac{m}{2\bar{r}}\right)^4 \left(d\bar{r}^2 + \bar{r}^2\, d\Omega^2\right)\,. \tag{1.51}$$

In these coordinates, there is only a singularity at $\bar{r} = 0$, which corresponds to $r = 0$.

As already indicated at the beginning of this paragraph, several experimental verifications of general relativity theory rest on the exterior Schwarzschild solution, namely, to mention only some catchwords,

- gravitational red shift,
- gravitational deflection of light ($\rightarrow$ gravitational lensing),
- general relativistic perihelion and periastron advance,
- time delay of Radar pulses (Shapiro effect).

Using additional structure from Einstein's theory, more predictions can be verified:

- Hulse-Taylor pulsar: emission of gravitational waves,
- Lense-Thirring effect (see Ciufolini et al. [2, 3] and Everitt [6]).

For more details on the experimental verification of Einstein's theory, compare Will [21].

1.5 Flat Minkowski spacetime, null coordinates, and Penrose diagram

In this section, we are going to analyze the Schwarzschild solution, in particular its singularity structure. For this purpose we will first have a look at null coordinates. The simplest testing ground in this context is the (flat) Minkowski space. Its metric, in Cartesian and spherical polar coordinates reads (c=1),

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = -dt^2 + dr^2 + r^2\, d\Omega^2 \,. \tag{1.52}$$

We define *advanced* and *retarded null coordinates* according to

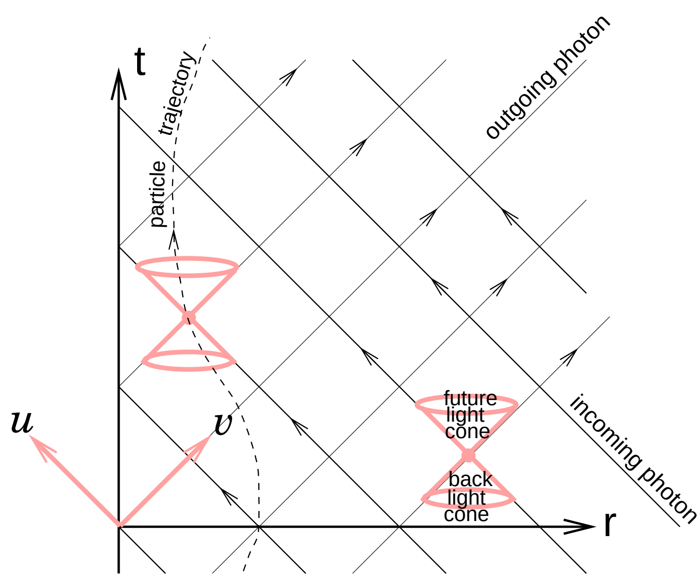
$$v := t + r\,, \quad u := t - r\,, \tag{1.53}$$

and find

$$ds^2 = -dv\, du + \frac{1}{4}\,(v-u)^2\, d\Omega^2 \,. \tag{1.54}$$

In figure 1.6 we show the Minkowski spacetime in terms of the new coordinates. Incoming photons, i.e., point-like particles with velocity $\dot{r} = -c = -1$, move on paths with $v = const.$ Correspondingly, we have for outgoing photons $u = const.$ The special relativistic wave-equation is solved by any function $f(u)$ and $f(v)$. The surfaces $f(u) = const.$ and $f(v) = const.$ represent the wavefronts which evolve with the velocity of light. The trajectory of every material particle with $v < c = 1$ has to remain inside the region defined by the surface $r = t$. In a (r,t)-diagram this surface is represented by a cone, the so-called *light cone.* Any point in the *future light cone* $r = t$ can be reached by a particle or signal with a velocity less than c. A given spacetime point P can be reached by a particle or signal from the spacetime region enclosed by the *back light cone* $r = -t$.

Figure 1.6. Minkowski spacetime in null coordinates



We can map, following Penrose, the infinitely distant points of spacetime into finite regions by means of a conformal transformation which leaves the light cones intact. Then we can display the whole infinite Minkowski spacetime on a (finite) piece of paper. Accordingly, introduce the new coordinates

$$\tilde{v} := \arctan v\,, \quad \tilde{u} := \arctan u\,, \qquad \text{for } -\pi/2 \le \tilde{v}\,, \tilde{u} \le +\pi/2\,. \tag{1.55}$$

Then the metric reads

$$ds^2 = \frac{1}{\cos^2\,\tilde{v}}\;\frac{1}{\cos^2\,\tilde{u}}\;\left[-d\tilde{v}\, d\tilde{u} + \frac{1}{4}\,\sin^2\,(\tilde{v}-\tilde{u})\, d\Omega^2\right]\,. \tag{1.56}$$

We can go back to time- and space-like coordinates by means of the transformation

$$\tilde{t} := \tilde{v} + \tilde{r}\,, \qquad \tilde{r} := \tilde{v} - \tilde{u}\,, \tag{1.57}$$

see (1.52). Then the metric reads,

$$ds^2 = \frac{-d\tilde{t}^2 + d\tilde{r}^2 + \sin^2 \tilde{r}\ d\Omega^2}{4\cos^2 \frac{\tilde{t}+\tilde{r}}{2}\ \cos^2 \frac{\tilde{t}-\tilde{r}}{2}}\,, \tag{1.58}$$

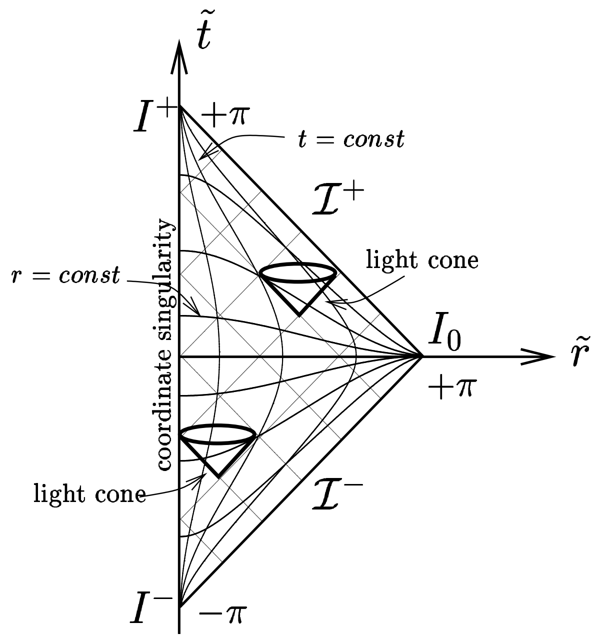
that is, up to the function in the denominator, it appears as a flat metric. Such a metric is called conformally flat (it is conformal to a static Einstein cosmos). The back-transformation to our good old Minkowski coordinates reads

$$t = \frac{1}{2}\left(\tan\frac{\tilde{t}+\tilde{r}}{2} + \tan\frac{\tilde{t}-\tilde{r}}{2}\right)\,, \tag{1.59}$$

$$r = \frac{1}{2}\left(\tan\frac{\tilde{t}+\tilde{r}}{2} - \tan\frac{\tilde{t}-\tilde{r}}{2}\right)\,. \tag{1.60}$$

Our new coordinates $\tilde{t}, \tilde{r}$ extend only over a finite range of values, as can be seen from (1.59), (1.60). Thus, in the Penrose diagram of a Minkowski spacetime, see figure 1.7, we can depict the whole Minkowski spacetime, with a coordinate singularity along $\tilde{r} = 0$. All trajectories of particles (with

Figure 1.7. Penrose diagram of Minkowski spacetime



velocity smaller than c) emerge form one single point, past infinity I^-, and all will eventually arrive at the one single point I^+, namely at future infinity. All incoming photons have their origin on the segment $\mathcal{I}^-$ (script I^- or "scri minus"), light-like past-infinity, and will run into the coordinate singularity on the $\tilde{t}$-axis . All outgoing photons arise from the coordinate singularity and cease on the line $\mathcal{I}^+$, light-like future infinity ("scri plus"). The entire spacelike infinity is mapped into the single point I^0.

Now, we have a really compact picture of the the Minkowski space. Next, we would like to proceed along similar lines in order to obtain an analogy for the Schwarzschild spacetime.

1.6 Schwarzschild spacetime and Penrose-Kruskal diagram

In relativity, light rays, the quasi-classical trajectories of photons, are null geodesics. In special relativity, this is quite obvious, since in Minkowski space the geodesics are straight lines and "null" just means $v = c$. A more rigorous argument involves the solution of the Maxwell equations for the vacuum and the subsequent determination of the normals to the wave surface (rays) which turn out to be null geodesics. This remains valid in general relativity. Null geodesics can be easily obtained by integrating the equation $0 = ds$. We find for the Schwarzschild metric, specializing to radial light rays with $d\phi = 0 = d\theta$,

$$t = \pm\left(r + 2m\,\ln\left|\frac{r}{2m} - 1\right|\right) + const\,. \tag{1.61}$$

If we denote with r_0 the solution of the equation $r+2m\ln\left|\frac{r}{2m}-1\right| = 0$, we have for the t-coordinate of the light ray $t(r_0) =: v$. Hence, if $r = r_0$, we can use v to label light rays. In view of this, we introduce v and u

$$v := t + r + 2m\ln\left|\frac{r}{2m} - 1\right|\,, \tag{1.62}$$

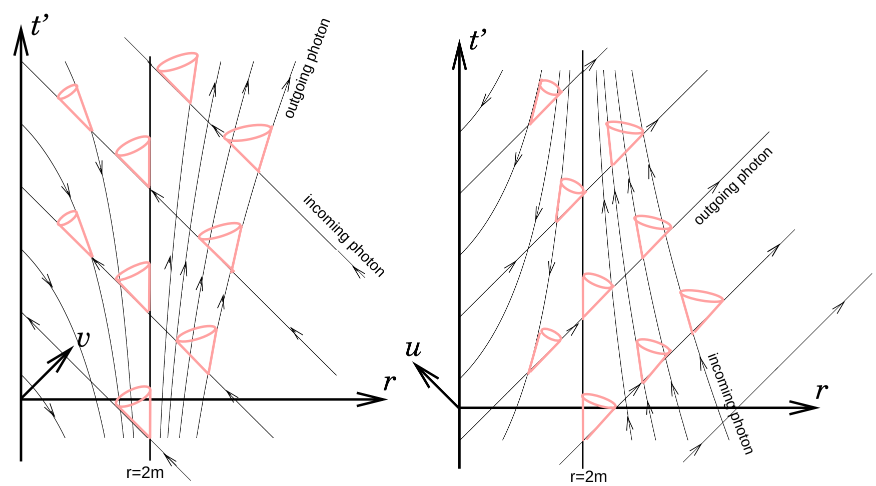
$$u := t - r - 2m\ln\left|\frac{r}{2m} - 1\right|\,. \tag{1.63}$$

Then ingoing null geodesics are described by $v = const$, outgoing ones by $u = const$, see figure 1.8. We define *ingoing Eddington-Finkelstein coordinates* by replacing the "Schwarzschild time" t by v. In these coordinates (v, r, θ, ϕ), the metric becomes

$$ds^2 = -\left(1 - \frac{2m}{r}\right)\,dv^2 + 2dv\,dr + r^2\,d\Omega^2\,. \tag{1.64}$$

For radial null geodesics $ds^2 = d\theta = d\phi = 0$, we find two solutions of (1.64), namely $v = const$ and $v = 4m\,\ln|r/2m - 1| + 2r + const$. The first one describes infalling photons, i.e., t increases if r approaches 0. At $r = 2m$, there is no singular behavior any longer for incoming photons. However, for outgoing photons, ingoing Eddington-Finkelstein coordinates are not well suited. Ingoing Eddington-Finkelstein coordinates are particular useful in order to describe the gravitational collapse. Analogously, for outgoing null geodesics take (u, r, θ, ϕ) as new coordinates. In these *outgoing Eddington-Finkelstein coordinates* the metric reads

Figure 1.8. In- and outgoing Eddington-Finkelstein coordinates (where we introduce t' with $v = t' + r$, $u = t' - r$).



$$ds^2 = -\left(1 - \frac{2m}{r}\right)\,du^2 - 2\,du\,dr + r^2\,d\Omega^2\,. \tag{1.65}$$

Outgoing light rays are now described by $u = const$, ingoing light rays by $u = -(4m\,\ln|r/2m - 1| + 2r) + const$. In these coordinates, the hypersurface $r = 2m$ (the "horizon") can be recognized as a null hypersurface (its normal is null or lightlike) and as a semi-permeable membrane.

Next we try to combine the advantages of in- and outgoing Eddington-Finkelstein coordinates in the hope to obtain a fully regular coordinate system of the Schwarzschild spacetime. Therefore we assume coordinates (u, v, θ, ϕ). Some (computer) algebra yields the corresponding representation of the metric:

$$ds^2 = -\left(1 - \frac{2m}{r(u,v)}\right) du\, dv + r^2(u,v)\, d\Omega^2 \,. \tag{1.66}$$

Unfortunately, we still have a coordinate singularity at $r = 2m$. We can get rid of it by reparametrizing the surfaces $u = const$ and $v = const$ via

$$\tilde{v} = \exp\left(\frac{v}{4m}\right) , \qquad \tilde{u} = -\exp\left(-\frac{u}{4m}\right) . \tag{1.67}$$

In these coordinates, the metric reads ($r = r(\tilde{u}, \tilde{v})$ is implicitly given by (1.67) and (1.63),(1.62))

$$ds^2 = -\frac{4r_S^3}{r(\tilde{u},\tilde{v})} \exp\left(-\frac{r(\tilde{u},\tilde{v})}{2m}\right) d\tilde{v}\, d\tilde{u} + r^2(\tilde{u},\tilde{v})\, d\Omega^2 \,. \tag{1.68}$$

Again, we go back from $\tilde{u}$ and $\tilde{v}$ to time- and space-like coordinates:

$$\tilde{t} := \frac{1}{2} \left(\tilde{v} + \tilde{u}\right) , \qquad \tilde{r} := \frac{1}{2} \left(\tilde{v} - \tilde{u}\right) . \tag{1.69}$$

In terms of the original Schwarzschild coordinates we have

$$\tilde{r} = \sqrt{\left|\frac{r}{2m} - 1\right|}\; \exp\left(\frac{r}{4m}\right) \cosh\frac{t}{4m} , \tag{1.70}$$

$$\tilde{t} = \sqrt{\left|\frac{r}{2m} - 1\right|}\; \exp\left(\frac{r}{4m}\right) \sinh\frac{t}{4m} . \tag{1.71}$$

The Schwarzschild metric

$$ds^2 = \frac{4r_S^3}{r} \exp\left(-\frac{r}{2m}\right) \left(-d\tilde{t}^{\,2} + d\tilde{r}^2\right) + r^2\, d\Omega^2 , \tag{1.72}$$

in these *Kruskal-Szekeres* coordinates $(\tilde{t}, \tilde{r}, \theta, \phi)$, behaves regular at the gravitational radius $r = 2m$. If we substitute (1.72) into the Einstein equation (via computer algebra), then we see that it is a solution of it for all $r > 0$. Equations (1.70), (1.71) yield

$$\tilde{r}^2 - \tilde{t}^{\,2} = \left|\frac{r}{2m} - 1\right| \exp\left(\frac{r}{2m}\right) . \tag{1.73}$$

Thus, the transformation is valid only for regions with $|\tilde{r}| > \tilde{t}$. However, we can find a set of transformations which cover the entire $(\tilde{t}, \tilde{r})$-space. They are valid in different domains, indicated here by I, II, III, and IV, to be explained below:

$$(\mathrm{I})\begin{cases} \tilde{t} &= \sqrt{\frac{r}{2m} - 1}\; \exp\left(\frac{r}{4m}\right) \sinh\frac{t}{4m} \\ \tilde{r} &= \sqrt{\frac{r}{2m} - 1}\; \exp\left(\frac{r}{4m}\right) \cosh\frac{t}{4m} \end{cases} \tag{1.74}$$

$$(\mathrm{II})\begin{cases} \tilde{t} &= \sqrt{1 - \frac{r}{2m}}\; \exp\left(\frac{r}{4m}\right) \cosh\frac{t}{4m} \\ \tilde{r} &= \sqrt{1 - \frac{r}{2m}}\; \exp\left(\frac{r}{4m}\right) \sinh\frac{t}{4m} \end{cases} \tag{1.75}$$

$$(\mathrm{III})\begin{cases} \tilde{t} &= -\sqrt{\frac{r}{2m} - 1}\; \exp\left(\frac{r}{4m}\right) \sinh\frac{t}{4m} \\ \tilde{r} &= -\sqrt{\frac{r}{2m} - 1}\; \exp\left(\frac{r}{4m}\right) \cosh\frac{t}{4m} \end{cases} \tag{1.76}$$

$$(\mathrm{IV})\begin{cases} \tilde{t} &= -\sqrt{1 - \frac{r}{2m}}\; \exp\left(\frac{r}{4m}\right) \cosh\frac{t}{4m} \\ \tilde{r} &= -\sqrt{1 - \frac{r}{2m}}\; \exp\left(\frac{r}{4m}\right) \sinh\frac{t}{4m} \end{cases} \tag{1.77}$$

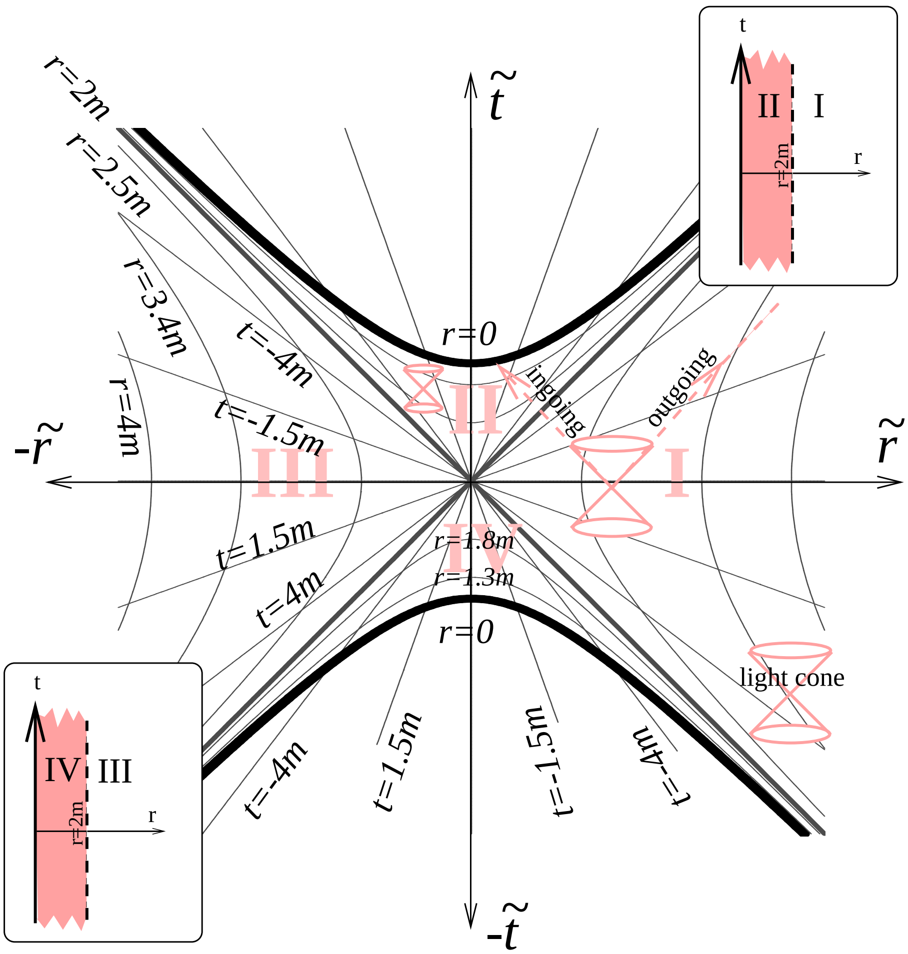
The inverse transformation is given by

$$\left(\frac{r}{2m} - 1\right) \exp\left(\frac{r}{2m}\right) = \tilde{r}^2 - \tilde{t}^{\,2} , \tag{1.78}$$

$$\frac{t}{4m} = \begin{cases} \operatorname{Artanh} \tilde{t}/\tilde{r} , & \text{for (I) and (III)} , \\ \operatorname{Artanh} \tilde{r}/\tilde{t} , & \text{for (II) and (IV)} . \end{cases} \tag{1.79}$$

The Kruskal-Szekeres coordinates $(\tilde{t}, \tilde{r}, \theta, \phi)$ cover the entire spacetime. By means of the transformation equations we recognize that we need two Schwarzschild coordinate systems in order to cover the same domain. Regions (I) and (III) both correspond each to an asymptotically flat universe with $r > 2m$. Regions (II) and (IV) represent two regions with $r < 2m$. Since $\tilde{t}$ is a time coordinate, we see that the regions are time reversed with respect to each other. Within these regions, real physical singularities (corresponding to $r = 0$) move along the lines $\tilde{t}^{\,2} - \tilde{r}^2 = 1$. From the form of the metric we can infer that the light-like geodesics (and therewith the light cones $ds = 0$) are lines with slope 1/2. This makes the discussion of the causal structure particularly simple.

Figure 1.9. Kruskal-Szekers diagram of the Schwarzschild spacetime



Finally, we would like to represent the Schwarzschild spacetime in a manner analogous to the Penrose diagram of the Minkowski spacetime. To this end, we proceed along the same line as in the Minkowskian case. First, we switch again to null-coordinates $v' = \tilde{t} + \tilde{r}$ and $u' = \tilde{t} - \tilde{r}$ and perform a conformal transformation which maps infinity into the finite (again, by means of the tangent function). Finally we return to a time-like coordinate $\hat{t}$ and a space-like coordinate $\hat{r}$. We perform these transformations all in one according to

$$\tilde{t} + \tilde{r} = \tan \frac{\hat{t} + \hat{r}}{2}, \tag{1.80}$$

$$\tilde{t} - \tilde{r} = \tan \frac{\hat{t} - \hat{r}}{2}. \tag{1.81}$$

The Schwarzschild metric then reads

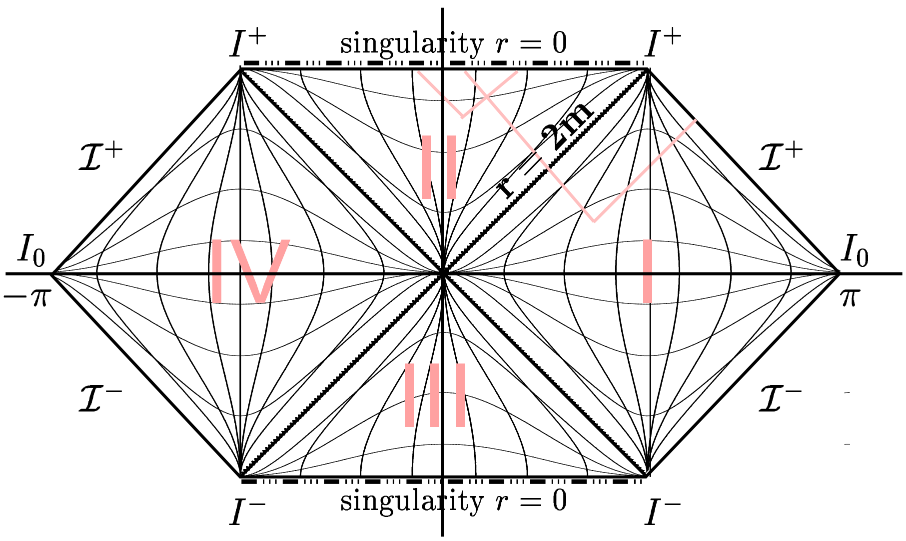
$$ds^2 = \frac{r_S^3}{r(\hat{r}, \hat{t})} \frac{\exp\left(-\frac{r(\hat{r},\hat{t})}{2m}\right) \left(-d\hat{t}^{\,2} + d\hat{r}^2\right)}{\cos^2 \frac{\hat{t}+\hat{r}}{2} \, \cos^2 \frac{\hat{t}-\hat{r}}{2}} + r^2(\hat{t}, \hat{r})\, d\Omega^2, \tag{1.82}$$

where the function $r(\hat{t}, \hat{r})$ is implicitly given by

$$\left(\frac{r}{2m} - 1\right) \exp\left(\frac{r}{2m}\right) = \tan\frac{\hat{t}+\hat{r}}{2}\,\tan\frac{\hat{t}-\hat{r}}{2}\,. \tag{1.83}$$

The corresponding Penrose-Kruskal diagram is displayed in figure 1.10.

Figure 1.10. Penrose-Kruskal diagram of the Schwarzschild spacetime



1.7 The interior Schwarzschild solution and the TOV equation

In the last section we investigated the gravitational field outside a spherically symmetric mass-distribution. Now its time time to have a look inside matter, see Adler, Bazin, and Schiffer [1]. Of course, in a first attempt, we have to make decisive simplifications on the internal structure of a star. We will consider cold catalyzed stellar material during the later phase of its evolution which can be reasonably approximated by a perfect fluid. The typical mass densities are in the range of $\approx 10^7\,\mathrm{g/cm^3}$ (white dwarfs) or $\approx 10^{14}\,\mathrm{g/cm^3}$ (neutron stars, i.e., pulsars). In this context we assume vanishing angular momentum.

We start again from a static and spherically symmetric metric

$$ds^2 = -\mathrm{e}^{A(r)}\,c^2\,dt^2 + \mathrm{e}^{B(r)}\,dr^2 + r^2\,d\Omega^2 \tag{1.84}$$

and the energy-momentum tensor

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu\, u_\nu + p\, g_{\mu\nu}\,, \tag{1.85}$$

where $\rho = \rho(r)$ is the spherically symmetric mass density and $p = p(r)$ the pressure (isotropic stress). This has to be supplemented by the equation of state which, for a simple fluid, has the form $p = p(\rho)$.

We compute the non-vanishing components of the field equation by means of computer algebra as (here $()' = d/dr$)

$$-\mathrm{e}^B \kappa r^2 c^2 \rho + \mathrm{e}^B + B'r - 1 = 0\,, \tag{1.86}$$

$$-\mathrm{e}^B \kappa p r^2 - \mathrm{e}^B + A'r + 1 = 0\,, \tag{1.87}$$

$$-4\mathrm{e}^B \kappa p r + 2A''r + (A')^2 r - A'B'r + 2A' - 2B' = 0\,. \tag{1.88}$$

The (ϕ,ϕ)-component turns out to be equivalent to the (θ,θ)-component. For convenience, we define a *mass function* $m(r)$ according to

$$\mathrm{e}^{-B} =: 1 - \frac{2m(r)}{r}\,. \tag{1.89}$$

We can differentiate (1.89) with respect to r and find, after substituting (1.86), a differential equation for $m(r)$ which can be integrated, provided $\rho(r)$ is assumed to be known

$$m(r) = \int_0^r \frac{\kappa}{2}\,\rho c^2\,\xi^2\,d\xi\,. \tag{1.90}$$

Differentiating (1.87) and using all three components of the field equation, we obtain a differential equation for A:

$$A' = -\frac{2p'}{p+\rho c^2}\,. \tag{1.91}$$

We can derive an alternative representation of A' by substituting (1.89) into (1.87). Then, together with (1.91), we arrive at the *Tolman-Oppenheimer-Volkoff* (TOV) equation

$$p' = -\frac{(\boldsymbol{\rho c^2} + p)(\mathbf{m} + \kappa\,pr^3/2)}{\mathbf{r}(\mathbf{r} - 2m)}\,. \tag{1.92}$$

The Newtonian terms are denoted by boldface letters. The system of equations consisting of (1.90), (1.91), the TOV equation (1.92), and the equation of state $p = p(\rho)$ forms a complete set of equations for the unknown functions $A(r)$, $\rho(r)$, $p(r)$, and $m(r)$, with

$$ds^2 = -\mathrm{e}^{A(r)}\,c^2\,dt^2 - \frac{dr^2}{1-\frac{2m(r)}{r}} - r^2\,d\Omega^2\,. \tag{1.93}$$

These differential equations have to be supplemented by initial conditions.

In the center of the star, there is, of course, no enclosed mass. Hence we demand $m(0) = 0$. The density has to be finite at the origin, i.e. $\rho(0) = \rho_c$, where ρ_c is the density of the central region. At the surfaces of the star, at $r = R_\odot$, we have to match matter with vacuum. In vacuum, there is no pressure which requires $p(R_\odot) = 0$. Moreover, the mass function should then yield the total mass of the star, $m(R_\odot) = M$. Finally, we have to match the components of the metric. Therefore, we have to demand $\exp[A(r_0)] = 1 - 2m(R_\odot)/R_\odot$.

Equations (1.86), (1.87), (1.86) and certain regularity conditions which generalize our boundary conditions, i.e.

- Regularity of the geometry at the origin,
- finiteness of central pressure and density,
- positivity of central pressure and density,
- positivity of of pressure and density,
- monotonic decrease of pressure and density,

impose conditions on the functions ρ and p. Then, even without the explicit knowledge of the equation of state, the general form of the metric can be determined. For most recent work, see Rahman and Visser [16] and the literature given there.

We can obtain a simple solution, if we assume a constant mass density

$$\rho = \rho(r) = const. \tag{1.94}$$

One should mention here that ρ is not the physically observable fluid density, which results from an appropriate projection of the energy-momentum tensor into the reference frame of an observer. Thus, this model is not as unphysical as it may look at the first. However, there are serious but more subtle objections which we will not discuss further in this context.

When $\rho = const$, we can immediately integrate (1.89) and thus obtain the metric component $\exp(B)$. Also (1.91) can be integrated. Then, after some more elementary integrations, we can make use of the boundary conditions. Finally, we arrive at the *interior & exterior Schwarzschild solution* for a spherically symmetric body [20]

$$ds^2 = \begin{cases} -\left(\frac{3}{2}\sqrt{1-\frac{R_\odot^2}{\hat{R}^2}} - \frac{1}{2}\sqrt{1-\frac{r^2}{\hat{R}^2}}\right)^2 c^2\,dt^2 + \frac{1}{1-\frac{r^2}{\hat{R}^2}}\,dr^2 + r^2 d\Omega^2\,, & r \le R_\odot\,, \\ -\left(1-\frac{2m}{r}\right)c^2\,dt^2 + \frac{1}{1-\frac{2m}{r}}\,dr^2 + r^2\,d\Omega^2\,, & r > R_\odot\,, \end{cases} \tag{1.95}$$

with

$$\hat{R} := \sqrt{\frac{3}{\kappa\rho c^2}}\,, \quad \rho = const\,. \tag{1.96}$$

For the sun we have $M_\odot \approx 2\times 10^{30}\,\mathrm{kg}$, $R_\odot \approx 7\times 10^{8}\,\mathrm{m}$ and subsequently $\rho_\odot \approx 1,4\times 10^{3}\,\mathrm{kg/m^3}$. This leads to $\hat{R}\approx 3\times 10^{11}$, that is, the radius of the star $R_\odot$ is much smaller than $\hat{R}$: $R_\odot < \hat{R}$. Hence the square roots in (1.95) remain real.

Visualization and comparison with a "Newtonian" star

From the continuous matching of the g_{rr}-component we can derive the relation $1 - 2M/R_\odot = 1 - R_\odot^2/\hat{R}^2$. Together with the definition of the Schwarzschildradius we find for the total gravitating mass of the star

$$M = \frac{4\pi}{3}\,R_\odot^3\,\rho\,. \tag{1.97}$$

Another method to obtain the total mass is to multiply the density ρ by the spatial volume of the star at a given time t_0. However, the total mass calculated that way is *larger* than the total gravitating mass (1.97). This is due to the fact that not mass (that is "rest-mass") alone but mass-energy gravitates. The negative gravitational binding forces thus contribute to the gravitating mass which appears in the metric.

Finally, some words about the geometry of the Schwarzschild spacetime. We can visualize its structure by means of an embedding in the following way: In the equatorial plane $\vartheta = \pi/2$ at a prescribed time $t = t_0$, the metric reads ($\hat{R}^2 = R_\odot^3/2m$)

$$ds^2 = \begin{cases} \left(1 - \frac{2m\,r^2}{R_\odot^3}\right)^{-1} dr^2 + r^2\,d\varphi^2 & \text{for} \quad r \le R_\odot\,, \\ \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\,d\varphi^2 & \text{for} \quad r > R_\odot\,. \end{cases} \tag{1.98}$$

These metrics are equivalent to 2D metrics induced by the 3D *euclidean* metric on a sphere or a hyperboloid, respectively. The 3D Euclidean metric is $ds^2 = dr^2 + r^2\,d\phi^2 + dz^2$. A surface rotationally symmetric around the z-axis is described by a parametrization $z = z(r)$. The metric induced on this surface is $ds^2 = [1 + (dz/dr)^2]\,dr^2 + r^2\,d\phi^2$. By comparison with the metrics above, we extract differential equations for $z(r)$ which can be easily solved. At $r = R_\odot$, the surfaces are continuously joined.

Outside, we have the usual vacuum Schwarzschild geometry which was discussed extensively in the previous section. We may add a few remarks. Obviously, a circle (or sphere, respectively) around the origin has a circumference of $2\pi r$, where r is the radial Schwarzschild coordinate. We also observe that the proper distance measured by a freely falling observer (who, in our picture, moves radially on the hyperboloid) is larger than the coordinate distance Δr. Inside the star we have the 3-geometry of a sphere with radius $\hat{R}$. Far away from the star we find flat Euclidean geometry.

The structure of this 3-geometry resembles the Newtonian case. Inside, we have a conformally flat space, where the Weyl ("tracefree part of the curvature") vanishes and the Ricci tensor is proportional to the mass-energy density. In the Newtonian case, the trace of the tidal matrix (the analogy to curvature) is proportional to the mass density, and, subsequently, its tracefree part vanishes. Outside, in vacuum, it is the other way around. There the trace parts are zero ($K_{aa} = 0$ and $\mathrm{Ric}_{\alpha\beta} = 0 = R$). The Newtonian tidal acceleration matrix is trace-free and reads

$$K_{ab} = \frac{GM}{r^3}\,\mathrm{diag}(1,1,-2)\,. \tag{1.99}$$

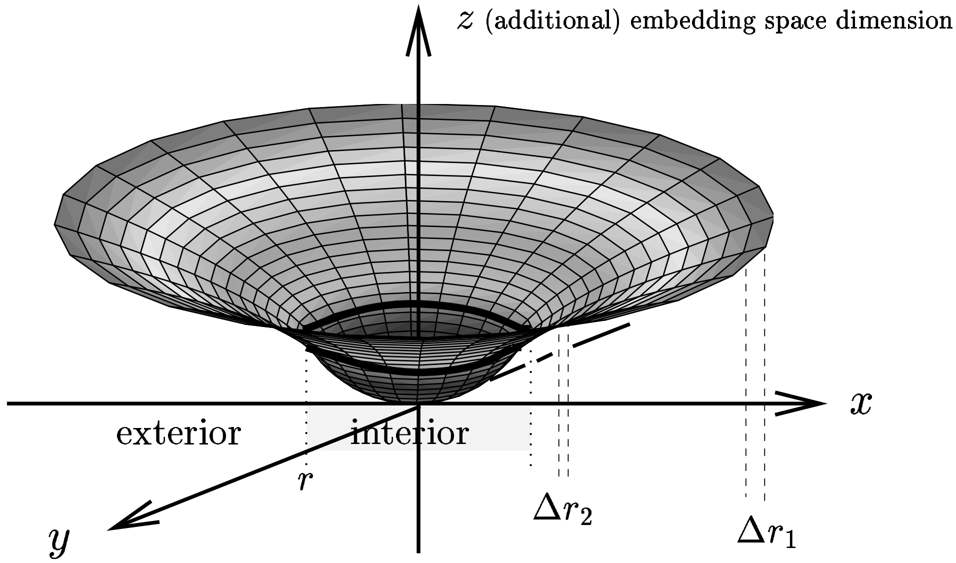
In Einstein's theory we have to use the equation for the geodesic deviation (1.39) in order to calculate the relative acceleration of two freely falling test particles. For the comoving observer, with $u^\alpha = (c,0,0,0)$, we find

$$\ddot{v}^\mu \stackrel{*}{=} c^2\,R^\mu{}_{0\nu 0}\,v^\nu = \frac{GM}{r^3}\,\mathrm{diag}(1,1,-2)\,(v^1,v^2,v^3)\,. \tag{1.100}$$

Thus, in a special frame, we have the same tidal accelerations as in the Newtonian case.

Accordingly, the gravitational field of a spherically symmetric body in Newton's 3D theory is very naturally embedded into Einstein's 4D theory.

Figure 1.11. Geometry of Schwarzschild spacetime



1.8 Computer algebra

As a typical example, we will demonstrate of how to obtain the exterior Schwarzschild solution by means of the computer algebra system Reduce and its package Excalc for applications in differential geomtery.

When Reduce is called, it prompts the user for input. Each statement has to be terminated by a semicolon. After pressing the return-key, the computer (hopefully) will produce a result. So lets start by loading the package Excalc:

```
load excalc ;
```

Then, we define the metric. Therefore we first introduce the functions which enter the ansatz for the metric λ and ν:

```
pform{nu,lamb}=0;
```

and declare on which variables they depend:

```
fdomain nu = nu(r,t), lamb=lamb(r,t) ;
```

Subsequently, we define the coframe and the metric:

```
coframe  o(t) = d t ,
         o(r) = d r ,
         o(theta) = d theta ,
         o(phi) = d phi
with metric g =  - exp(nu) * o(t) * o(t)
                 + exp(lamb) * o(r) * o(r)
                 + r**2 * o(theta) * o(theta)
                 + r**2 *sin(theta)**2 * o(phi) * o(phi) ;
```

In Excalc, which is a package designed to allow for calculations using the calculus of exterior differential forms, it is convenient to compute partial derivatives of scalar functions according to

$$\partial_\alpha \phi = e_\alpha \rfloor d\phi = e_\alpha \rfloor (\partial_\alpha \phi \, dx^\alpha) \ , \tag{1.101}$$

where we have introduced the vector basis e_α dual to the coframe, i.e. $e_\alpha \rfloor dx^\beta = \delta^\alpha_\beta$ ($\rfloor$ is the interior product). Accordingly, we compute the vector basis

```
frame e ;
```

and define the Christoffel symbol

```
pform chris(i,j,k) = 0 ;
index_symmetries chris(i,j,k): symmetric in {j,k}  ;

chris(-i,-j,-k) := (1/2) * (   e(-k)_|d g(-i,-j)
                             + e(-j)_|d g(-i,-k)
                             - e(-i)_|d g(-j,-k) );
```

Next, we compute the Riemannian curvature tensor. By means of the declaration `index_symmetries` we can explicitly implement the index symmetries of tensors which saves a lot of memory and computation time. Moreover, the printed output then encompasses automatically only independent components.

```
pform riem(i,j,k,l) = 0;
index_symmetries riem(i,j,k,l): antisymmetric in {i,j},{k,l}
                                    symmetric in {{i,j},{k,l}} ;

riem(i,-j,-k,-l) :=   e(-k)_|d chris(i,-j,-l)
                    - e(-l)_|d chris(i,-j,-k)
                    + chris(i,-m,-k) * chris(m,-j,-l)
                    - chris(i,-m,-l) * chris(m,-j,-k) ;
```

Then, we introduce Ricci tensor, curvature scalar and the Einstein tensor.

```
pform ricci(i,j)=0 ;
ricci(-i,-j) := riem(k,-i,-k,-j) ;

pform rscalar = 0;
rscalar := ricci(-i,i) ;

pform einstein(i,j) = 0 ;
einstein(i,j) := ricci(i,j) - (1/2) * rscalar * g(i,j) ;
```

Now we implement the vacuum field equation:

```
pform zero(i,j) = 0 ;
zero(i,j) :=    kappa * einstein(i,j) + kosmo * g(i,j) ;
```

The next step is to look at the output and get some ideas how to proceed ... With a computer algebra system, we can manipulate very easily systems of equation in order to obtain new, more simple equations. By entering (`num` yields the numerator of a fraction):

```
0 = num(zero(t,t)) + num(zero(r,r)) ;
```

we get

$$0 = \partial_r \,\lambda\, \kappa\, r + \partial_r \,\nu\, \kappa\, r\,. \tag{1.102}$$

Accordingly, the sum $f := \lambda + \nu$ has to be independent of r and thus is function of t alone. Then we can perform a rescaling of the time coordinate

$$t \longrightarrow t' = \int dt\, \mathrm{e}^{-f(t)/2} \tag{1.103}$$

such that

$$dt' = \mathrm{e}^{-f(t)/2}\, dt \tag{1.104}$$

Hence, the ansatz for the metric does not change apart from the t,t-component:

$$\mathrm{e}^{\nu(r,t)}\, dt \to \mathrm{e}^{\nu(r,t')-f(t')} dt'^2 =: \mathrm{e}^{\nu'}\, dt'^2 \tag{1.105}$$

or

$$\nu = \nu' + f(t) \tag{1.106}$$

and thus

$$\lambda = -\nu' \,. \tag{1.107}$$

Eventually, we can set:

```
lamb := - nu ;
```

and suppress the dashes from now on. Next, we notice that

$$0 = \mathtt{zero}(\mathtt{r},\mathtt{t}) = -\frac{\kappa\,\partial_t \nu}{\mathrm{e}^{2\,\nu}\,r} \tag{1.108}$$

Consequently, the function ν can not depend on t. We take this into account by substituting:

```
@(nu,t) := 0 ;
```

For convenience, we get rid of the exp-functions:

```
pform psi = 0 ;
fdomain psi = psi(r) ;
nu := log(psi) ;

zero(i,j) := zero(i,j);
```

The r,r-component of the field equation can be solved for $\partial_r \psi$. We can do this with the computer by means of the `solve` operator

```
solve(zero(r,r)=0,@(psi,r)) ;
```

We then substitute the result into the field equation

```
@(psi,r) := ( -kappa*psi + kappa - kosmo*r**2)/(kappa*r);
```

It turns out that then all components of the field equations are fulfilled already. There remains the task to solve the differential equation

$$\partial_r\,\psi + \frac{\psi}{r} - \frac{1}{r} + \frac{\Lambda}{\kappa}\,r \;= 0 \tag{1.109}$$

We may solve this ordinary differential equation by means of an appropriate package like the Reduce package `odesolve`

```
load odesolve ;
odesolve(df(psi,r)-@(psi,r),psi,r) ;
```

By setting the integration constant to $-2m$ we finally arrive at:

$$\psi = 1 - \frac{2m}{r} + \frac{\Lambda}{3}\,r^2 \,. \tag{1.110}$$

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