Black Holes

A Cursory Introduction to General Relativity

A Little History

- At the turn of the 20th century, the laws of electrodynamics and mechanics contradicted each other.
- Galiean mechanics contained no reference to the speed of light, but Maxwell’s equations and experiments said that light goes at the speed of light no matter how fast you are going.

Lorentz transformations

- To deal with this people argued that there should be new rules to add velocities and that the results of measuring an object’s mass or length as it approaches the speed of light would defy one’s expectations.

Enter Einstein (1)

- Einstein argued that the constancy of the speed of light was a property of space(time) itself.
  - The Newtonian picture was that everyone shared the same view of space and time marched in lockstep for everybody.
  - So, people would agree on the length of objects and the duration of time between events $dl^2 = dx^2 + dy^2 + dz^2, dt$

Enter Einstein (2)

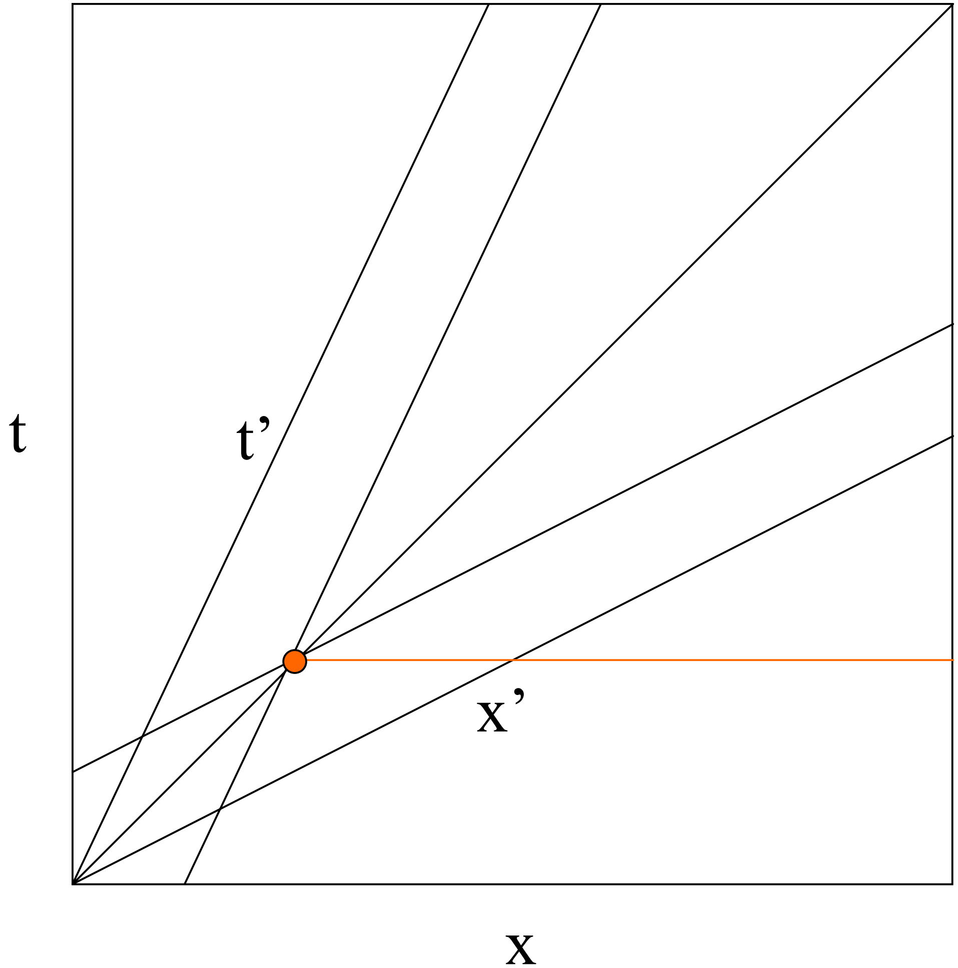
- Einstein argued that spacetime was the important concept and that the interval between events was what everyone could agree on. $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$
- This simple idea explained all of the nuttiness that experiments with light uncovered but it also cast the die for the downfall of Newtonian gravity.

Exit Newton (1)

- Newtonian gravity was action at a distance (Newton himself wasn't happy about this). This means that if you move a mass, its gravitational field will change everywhere instantaneously.
- In special relativity this leads to contradictions.

Exit Newton (2)

- At the event marked by the circle, a mass is shaken, the gravitational field will change instantly along red line.
- Someone moving relative to the mass will find that the field changes before the mass is moved.
- This is bad, bad, bad.



Einstein again

- Instead of trashing the brand-new special theory of relativity, Einstein decided to rework the venerable theory of gravity. He came upon the general theory of relativity.

A First Try (1)

- Newtonian gravity looks a lot like electrostatics: $\nabla^2\phi = 4\pi G\rho$
- Let's generalize it as a relativistic scalar field: $\nabla^2\phi - c^2\frac{d^2\phi}{dt^2} = 4\pi G\rho$
- What is ρ? The mass (or energy) density that one measures depends on velocity but the L.H.S. does not, so this equation is no Lorentz invariant.

A First Try (2)

- The relativistic generalization of the mass or energy density is the energy-momentum tensor. For a perfect fluid you have, $T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right) u^\alpha u^\beta + p g^{\alpha\beta}$ where $g^{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$ is the metric tensor (N.B. this is the other convention).
- We can get a scalar by taking

$$T = g_{\alpha\beta} T^{\alpha\beta} = \rho c^2 - 3p$$

$$\nabla^2 \phi - c^2 \frac{d^2\phi}{dt^2} = 4\pi \frac{G}{c^2} T$$

A First Try (3)

- The energy-momentum tensor of an electromagnetic field is traceless so $T=0$.
- This means that photons or the energy in a electric field does not generate gravity.
- This is bad, bad, bad.
- If photons feel gravity, momentum is not conserved.
- If photons don't feel gravity, energy is not conserved.

What to do?

- The next obvious step would be a vector field like electromagnetism, but it isn't obvious how to make a vector from the energy-momentum tensor.
- How about a tensor field? So

$$\Box^2 h^{\alpha\beta} = 4\pi \frac{G}{c^2} T^{\alpha\beta}$$

- But we would like gravitational energy to gravitate, so *h* should be on both sides.

Einstein's solution

- Einstein assumed that all objects follow the same paths in a gravitational field regardless of their mass or internal composition (strong equivalence principle), so he suggested that the metric itself ($g^{\alpha\beta}$) should play the role of *h* and objects follow extremal paths through the spacetime described by $g^{\alpha\beta}$.

The Geodesic Equation (1)

- Let's make some definitions:

$$u^\alpha = \frac{dx^\alpha}{ds}, u_\alpha = g_{\alpha\beta}u^\beta \text{ and } g_{\alpha\beta,\gamma} = \frac{dg_{\alpha\beta}}{dx^\gamma}$$

- Using the definition of the metric

$$\begin{aligned}\delta(ds^2) &= 2ds\delta(ds) = \delta(g_{\alpha\beta}dx^\alpha dx^\beta) \\ &= dx^\alpha dx^\beta g_{\alpha\beta,\gamma}\delta x^\gamma + 2g_{\alpha\beta}dx^\alpha d(\delta x^\beta)\end{aligned}$$

- Solving for δ(*ds*) yields

$$\begin{aligned}\delta s &= \int \delta(ds) = \int \left(\frac{1}{2}u^\alpha u^\beta g_{\alpha\beta,\gamma} \ \delta x^\gamma + g_{\alpha\beta}u^\alpha \frac{d\delta x^\beta}{ds} \right) ds \\ &= \int \left[\frac{1}{2}u^\alpha u^\beta g_{\alpha\beta,\gamma} \ \delta x^\gamma - \frac{d}{ds}\left(g_{\alpha\beta}u^\alpha\right) \delta x^\beta \right] ds\end{aligned}$$

The Geodesic Equation (2)

- Because the variation is arbitrary we can set its coefficient equal to zero:

$$\frac{1}{2}u^\alpha u^\beta g_{\alpha\beta,\gamma} - \frac{d}{ds}(g_{\alpha\gamma}u^\alpha) = 0$$

$$\frac{1}{2}u^\alpha u^\beta g_{\alpha\beta,\gamma} - \frac{du_\gamma}{ds} = 0$$

$$\frac{1}{2}u^\alpha u^\beta g_{\alpha\beta,\gamma} - g_{\alpha\gamma}\frac{du^\alpha}{ds} - u^\alpha u^\beta g_{\alpha\gamma,\beta} = 0$$

$$-g_{\alpha\gamma}\frac{du^\alpha}{ds} + \frac{1}{2}u^\alpha u^\beta (g_{\alpha\beta,\gamma} - g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}) = 0$$

The Geodesic Equation (3)

- Finally we have $g_{\alpha\gamma}\frac{du^\alpha}{ds} + \Gamma_{\gamma,\alpha\beta}u^\alpha u^\beta = 0$

  where $\Gamma_{\gamma,\alpha\beta} = \frac{1}{2}(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma})$ is the connection coefficient or Christoffel symbol.
- We have defined new concept but does it show up elsewhere.

Tensors (1)

- The quantities like $T_{\alpha\beta}$ that we have been manipulating are called tensors, and they have special properties. Specifically they transform simply under coordinate transformations. $T_{\alpha\beta} = \frac{\partial x'^{\gamma} \partial x'^{\delta}}{\partial x^{\alpha} \partial x^{\beta}} T'_{\gamma\delta}$
- Also if the metric isn't constant you would expect derivatives to depend on how the coordinates change as you move too.

Tensors (2)

- We want a derivative that transforms like a tensor (this is also called the connection).
- The derivative of a scalar quality should be simple; it does not refer to any directions, so we define the covariant derivative to be $\phi_{;\alpha} = \phi_{,\alpha}$
- Let's assume that the chain and product rules work for the covariant derivative like the normal one that we are familiar with.

Tensors (3)

- Let's prove a result about the metric, the tensor that raises and lowers indices.

$$
\begin{aligned}
A_{\beta;\alpha} &= (g_{\beta\gamma} A^\gamma)_{;\alpha} \\
&= g_{\beta\gamma;\alpha} A^\gamma + g_{\beta\gamma} A^\gamma_{;\alpha} = g_{\beta\gamma;\alpha} A^\gamma + A_{\beta;\alpha} \\
g_{\beta\gamma;\alpha} &= 0
\end{aligned}
$$

Tensors (4)

- Let's calculate the covariant derivative of a vector norm. $\phi = g_{\alpha\beta} A^\alpha A^\beta$

$$\phi_{,\gamma} = g_{\alpha\beta,\gamma} A^\alpha A^\beta + 2 g_{\alpha\beta} A^\alpha_{\ ,\gamma} A^\beta$$

$$\phi_{;\gamma} = g_{\alpha\beta;\gamma} A^\alpha A^\beta + 2 g_{\alpha\beta} A^\alpha_{\ ;\gamma} A^\beta$$

- Remember that the first term on the right-hand side of the last line vanishes so

$$2 g_{\alpha\beta} A^\alpha_{\ ;\gamma} A^\beta = \left(g_{\alpha\beta,\gamma} A^\alpha + 2 g_{\alpha\beta} A^\alpha_{\ ,\gamma} \right) A^\beta$$

$$\text{but } A_{\beta;\gamma} = A_{\beta,\gamma} + \frac{1}{2} g_{\alpha\beta,\gamma} A^\alpha \text{ is not true.}$$

Tensors (5)

- With help from the previous result and the result on page (3) you find that

$$A^{\alpha}_{;\beta} = A^{\alpha}_{,\beta} + \Gamma^{\alpha}_{\gamma\beta} A^{\gamma}$$

$$A_{\alpha;\beta} = A_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta} A_{\gamma}$$

$$A^{\alpha\beta}_{;\gamma} = A^{\alpha\beta}_{,\gamma} + \Gamma^{\alpha}_{\delta\gamma} A^{\delta\beta} + \Gamma^{\beta}_{\delta\gamma} A^{\alpha\delta}$$

where

$$\Gamma^{\delta}_{\alpha\beta} = \frac{1}{2} g^{\delta\gamma} \left(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma} \right)$$

Some Important Tensors (1)

- First, we measure scalar quantities the length of one vector along the direction of another. These scalars do not depend on the coordinate system.
- Coordinate vectors - dx^α
- Four velocity and four momentum - u^α and p^α
- Killing vectors (ξ^α) hold the key to the symmetry of the spacetime. $\xi^\alpha p_\alpha$ is constant along a geodesic.

Some Important Tensors (2)

- I am moving with four-velocity u^α and I detect a particle with four-momentum p^α. I would measure an energy of $g_{\alpha\beta}\, u^\alpha p^\beta$.
- If there is an electromagnetic field $F_{\alpha\beta}$. I would measure an electric field of $u^\alpha F_{\alpha\beta}$ and a magnetic field of u^α (dual $F)_{\alpha\beta}$.
- Of course, $g_{\alpha\beta}$ is the most important tensor of all. Without it we could not construct scalars and measure anything.

A Second Look at the Geodesic Equation

- Now we know what the covariant derivative looks like, the geodesic equation looks very simple.

$$g_{\alpha\gamma}\frac{du^\alpha}{ds} + \Gamma_{\gamma,\alpha\beta}u^\alpha u^\beta = 0$$

$$\frac{du^\gamma}{ds} + \Gamma^\gamma_{\alpha\beta}u^\alpha u^\beta = 0$$

$$u^\beta u^\gamma_{\ ,\beta} + \Gamma^\gamma_{\alpha\beta}u^\alpha u^\beta = u^\beta u^\gamma_{\ ;\beta} = 0$$

Christoffels

- The Christoffel symbols are not a tensor.
- From the rule for tensor transformation if a tensor is zero in one coordinate system it will be zero in all others.
- If I use the geodesics themselves, I can set up a coordinate system *locallly* in which the Christoffels vanish.
- However, if the geodesics diverge the Christoffels won't be zero everywhere.

Curvature

- Riemann figured out a tensorial quantity that measures how the geodesics diverge from each other and traces the curvature of the space. It is the Riemann curvature tensor.

$$R^{\mu}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\alpha} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha}$$

- It is like the curl of the Christoffel.   It transforms as a tensor.

Curvature and Geodesics

- If you have two geodesics that are separated by an infinitesimal distance v^μ, the separation evolves as
$$\frac{D^2 v^\mu}{D\tau^2} = \dot{x}^\nu \dot{x}^\alpha v^\beta R^\mu_{\ \nu\alpha\beta}$$
- Compare with the tidal force of Newton
$$\frac{D^2 v_a}{D\tau^2} = v_b f_{a,b} = -v_b \phi_{,a,b}$$
- So Riemann in GR is like $\phi_{,a,b}$ for Newton.

Poisson Equation $\Rightarrow$ Einstein Equation

- The equation for the potential in Newtonian gravity is $\nabla^2\phi = \phi_{a,a} = 4\pi G\rho$
- Let's define two new quantities, the Ricci tensor and scalar: $R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$ and $R = R^{\alpha}_{\alpha}$
- Einstein's field equation is

$$\underbrace{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}}_{\text{Einstein Tensor } G_{\mu\nu}} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

How to solve the equation?

- For an analytic solution, you need a high degree of symmetry and a simple expression for the energy-momemtum tensor:
  - Static spherically symmetric: vacuum, perfect fluid, electric field, scalar field
  - Homogeneous: perfect fluid
  - Stationary Axisymmetric: vacuum, electric field
- Numerical solutions are also difficult because the equations are non-linear.

Spherically symmetric vacuum (1)

- Let's try to find a spherically symmetric solution without matter. It starts with a trial metric:

  $$ds^2 = e^{\nu} c^2 dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - e^{\lambda} dr^2$$

  This equation means $ds^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta}$
- The functions ν and λ depend on t and r.
- There could also be a *drdt* term but we could eliminate it by a coordinate transformation.

Spherically symmetric vacuum (2)

- With Maple we can quickly get the nonzero components of the Einstein tensor.

$$\frac{8\pi G}{c^4}T^0_{\ 0} = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2}$$

$$\frac{8\pi G}{c^4}T^0_{\ 1} = -e^{-\lambda}\frac{\dot{\lambda}}{r}$$

$$\frac{8\pi G}{c^4}T^1_{\ 1} = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}$$

$$\frac{8\pi G}{c^4}T^2_{\ 2} = \frac{8\pi G}{c^4}T^3_{\ 3} = \mathbb{UGLY}$$

Spherically symmetric vacuum (3)

- The left-hand sides equal zero, so the second equation tells us $\lambda(r)$.
- The sum of the first and third tell us that, $\lambda' + \nu' = 0$ so $\lambda + \nu = f(t)$. We can redefine the time coordinate such that $f(t) = 0$ so the metric is static (it is not a function of time).

Spherically symmetric vacuum (4)

- We can integrate the remaining equations to give $e^{-\lambda} = e^{\nu} = 1 + \frac{K}{r}$

$$ds^2 = \left(1 + \frac{K}{r}\right) c^2 dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - \left(1 + \frac{K}{r}\right)^{-1} dr^2$$

- Let's calculate u^α for a stationary observer at radius r and then the gravitational redshift from r to infinity.

$$u^0 = \frac{dt}{d\tau} = \left(1 + \frac{K}{r}\right)^{-1/2}, P_\infty = P\left(1 + \frac{K}{r}\right)^{-1/2}$$

$$E_\infty = E\left(1 + \frac{K}{r}\right)^{1/2}, \frac{\Delta E}{E} = \frac{K}{2r} = \frac{\phi}{c^2} = -\frac{GM}{rc^2}, K = -\frac{2GM}{c^2}$$

Schwarzschild solution (1)

- What happens at the Schwarzschild radius?

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

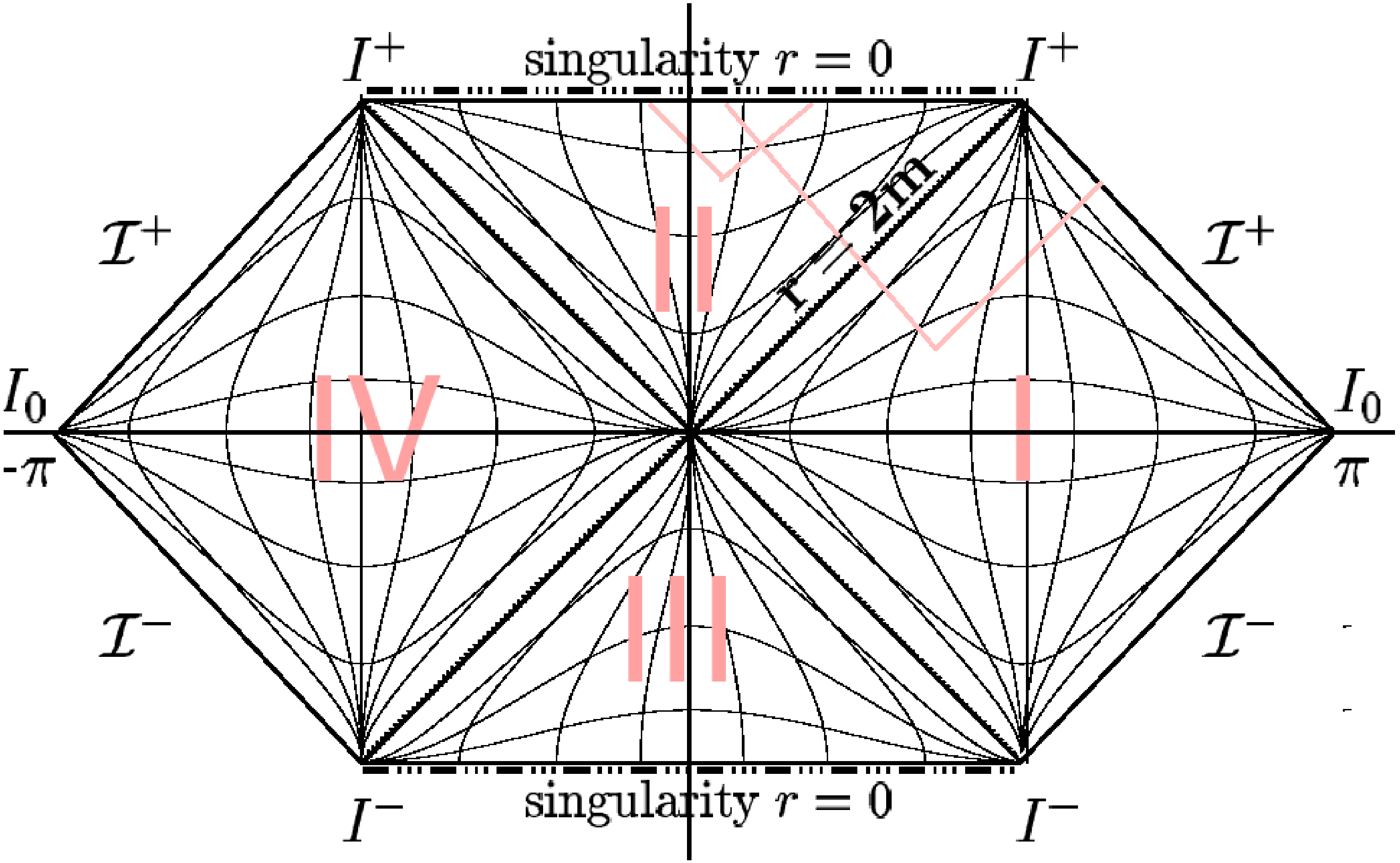
- A finite proper time lasts an infinite coordinate time. $u^0 = \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1/2}$
- What about the tidal forces (curvature)?

$$R^{\alpha\beta}_{\delta\gamma} = A\frac{M}{r^3}$$

Schwarzschild solution (2)

- Within the Schwarzschild radius the coefficient of dt^2 is negative (spacelike) while that of dr^2 is positive (timelike).
- As time passes for someone within the Schwarzschild radius she approaches $r=0$.
- One can use different sets of coordinates to illustrate that the Schwarzschild radius is not singular.

Penrose Diagram (1)



Penrose Diagram (2)

- In a Penrose diagram, each point represents a sphere in space. Light rays travel at 45°, timelike trajectories lie between the vertical and 45°. Infinity is also depicted.
- The Schwarzschild singularity is spacelike (like a wall). Once you are within $r = 2M$, you cannot avoid it.

A Rotating Black Hole

- The metric for a rotating black hole is somewhat more complicated.

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{4aMr\sin^2\theta}{\Sigma} dt\,d\phi - \frac{\Sigma}{\Delta} dr^2$$
$$- \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2$$

$$a = \frac{J}{M}, \Delta = r^2 - 2Mr + a^2, \Sigma = r^2 + a^2\cos^2\theta$$

- There are two horizons where $\Delta=0$.

$$r_\pm = M \pm \left(M^2 - a^2\right)^{1/2}$$

Something else special (1).

- Let's consider someone rotating around the hole with angular velocity $\Omega = u^\phi / u^t$ and no other velocity. Is there a limit to her angular velocity?

$$\left(\frac{ds}{d\tau}\right)^2 = \left(u^t\right)^2 \left(g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}\right)$$

The term in the brackets must be greater than zero because the LHS equals one.

Something else special (2).

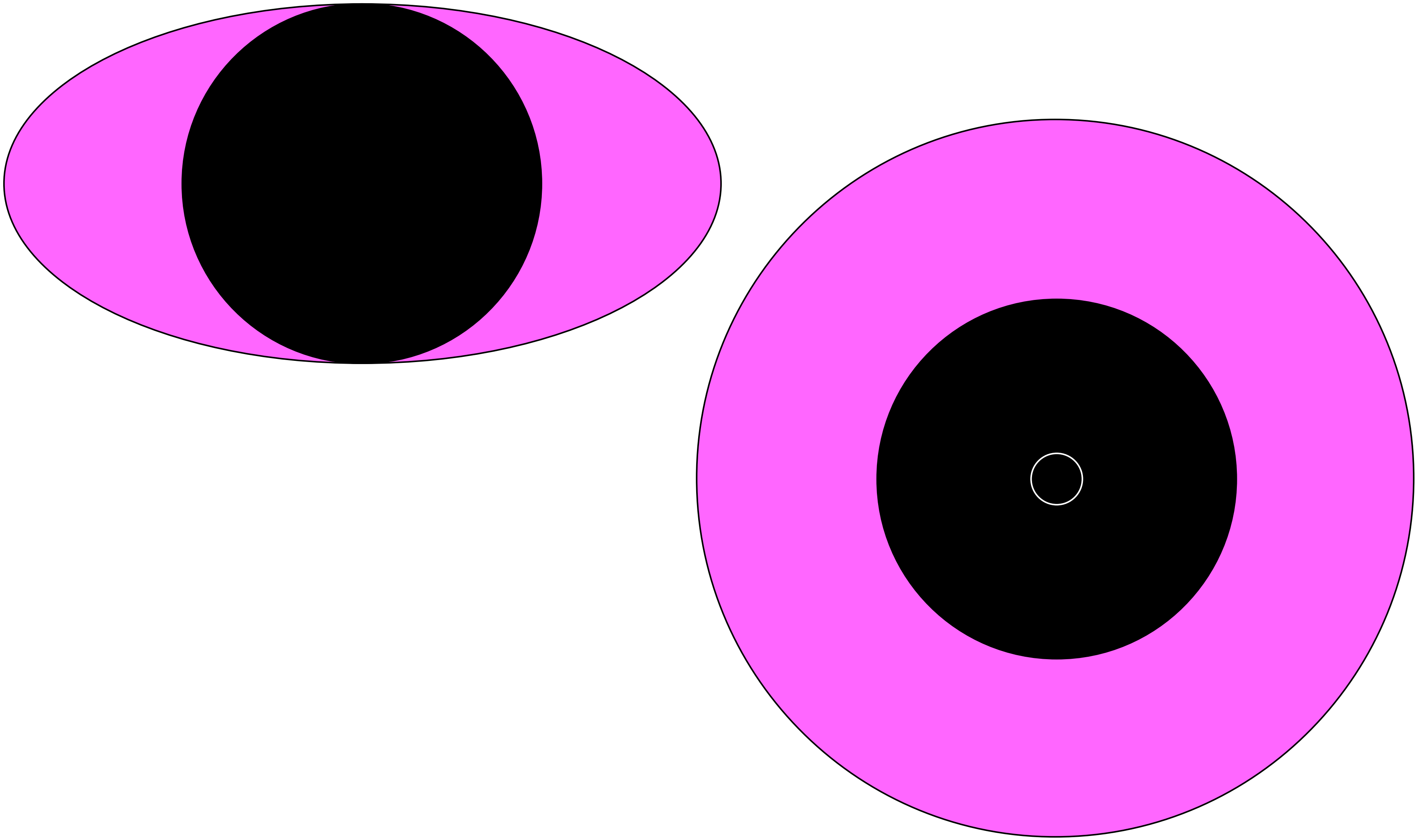
- Let's solve for the range in Ω.

$$\frac{-g_{t\phi}-\left(g_{t\phi}^{2}-g_{tt}g_{\phi\phi}\right)^{1/2}}{g_{\phi\phi}} < \Omega < \frac{-g_{t\phi}+\left(g_{t\phi}^{2}-g_{tt}g_{\phi\phi}\right)^{1/2}}{g_{\phi\phi}}$$

- If g_{tt}=0 then Ω_{min}=0, this occurs at

$$r_0 = M + \left(M^2 - a^2\cos^2\theta\right)^{1/2}$$

A Kerr Black Hole



Penrose Diagram

- The structure of the spacetime of a rotating black hole is much more complicated.
- The singularity is timelike (you can avoid it if you try), but r_- might be troublesome.