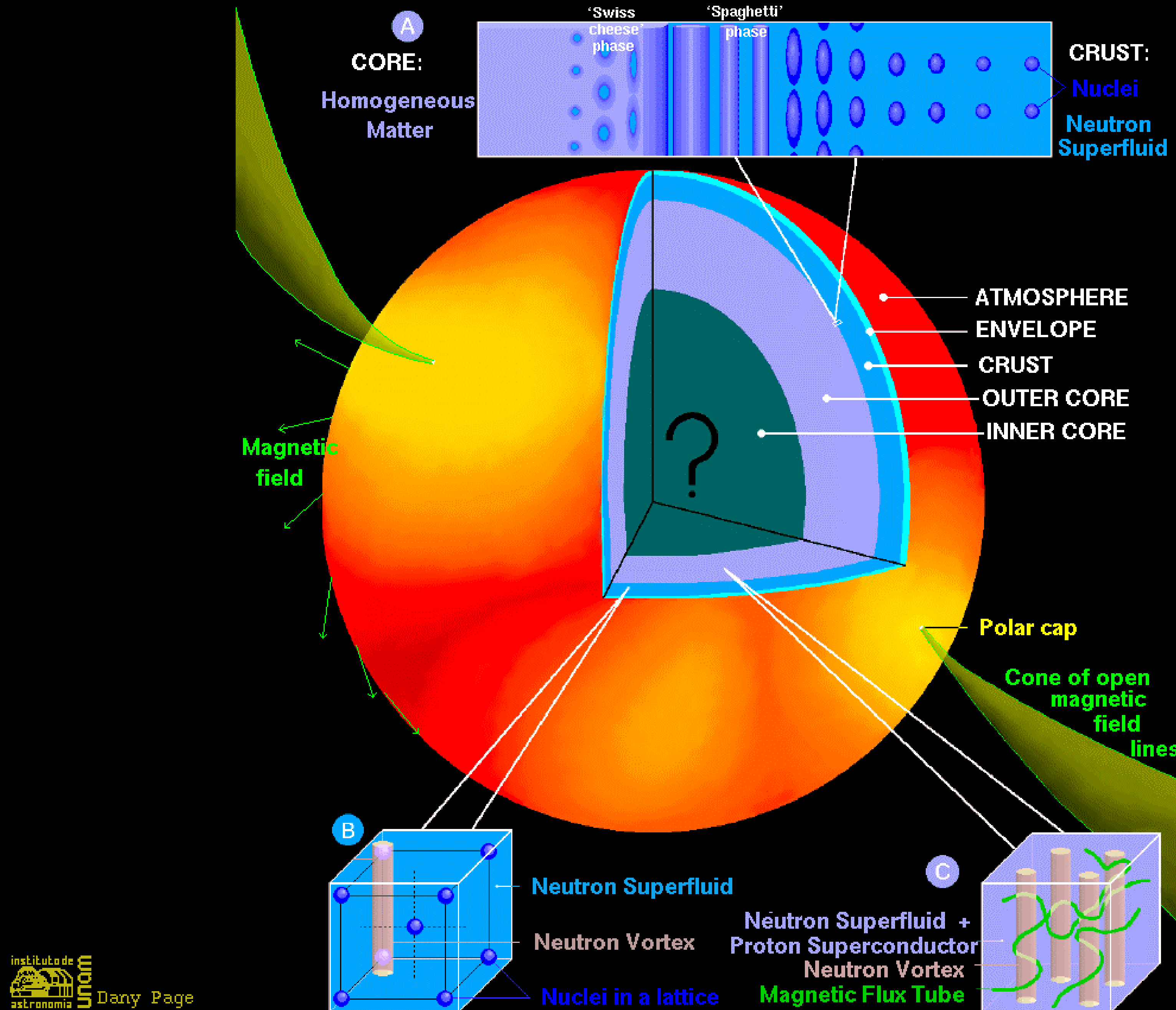
Structure of Neutron Stars

The Nuclear Equation of State and Other Things

A NEUTRON STAR: SURFACE and INTERIOR



What are the various regions?

- Atmosphere: the region near the stellar surface where most of the photons originate.   Only a few millimeters thick.
- Envelope: the surface region that throttles the heat flux (more on this next week): free electrons and nuclei, a metal. (sometimes called outer crust)
- Crust: free electrons, nuclei and free neutrons (sometimes called inner crust)

More regions

- Outer core: free neutrons, free protons, free electrons and other particles (no more nuclei)
- Inner core: dunno. It could be like the outer core or it could contain free quarks.
- In a “quark” or “strange” star, the core and inner crust consist of free quarks.

Relativistic Stellar Structure (1) - Equations

- OV (1939) give the equations of stellar structure in GR.

$$\frac{du}{dr} = 4\pi r^2 \rho(p)$$

$$\frac{dp}{dr} = -\frac{p + \rho(p)}{r(r - 2u)} \left(4\pi p r^3 + u\right)$$

ρ	Energy Density
p	Pressure
u	Enclosed **gravitational** mass
r	Circumferential radius

Relativistic Stellar Structure (2) - What's new?

- The relativistic equations of stellar structure are deceptively similar to the Newtonian results.

$$\frac{du}{dr} = 4\pi r^2 \rho(p)$$

$$\frac{dp}{dr} = -\frac{\boxed{p} + \rho(p)}{r(r \boxed{- 2u})} \left(\boxed{4\pi p r^3} + u \right)$$

New Bits

Relativistic Stellar Structure (3) - Nonlinearity

- Let's take $\rho \to \alpha\rho$ and see how the equations transform.

$$\frac{du}{dr} = 4\pi r^2 \alpha\rho(p) \quad \text{so} \quad u \to \alpha u \qquad \boxed{\text{New Bits}}$$

$$\frac{dp}{dr} = -\frac{\boxed{p} + \alpha\rho(p)}{r(r \boxed{-2\alpha u})}\left(\boxed{4\pi p r^3} + \alpha u\right)$$

- If it weren't for the new bits, we would have $p \to \alpha^2 p$, but the pressure generates more gravity. Even worse so does the gravity.

Relativistic Stellar Structure (4) - Nonlinearity

- The nonlinearity in the pressure is sufficient to transform a well-behaved solution into a singular one.
- The term in the denominator is even less benign. It defines a radius where the gravitational acceleration diverges.

$$\frac{du}{dr} = 4\pi r^2 \alpha\rho(p) \quad \text{so} \quad u \rightarrow \alpha u$$

$$\frac{dp}{dr} = -\frac{p + \alpha\rho(p)}{r(r - 2\alpha u)}\left(4\pi p r^3 + \alpha u\right)$$

New Bits

Relativistic Stellar Structure (5) - Solutions

- These nonlinear equations have a few non-trivial solutions, and one of them is pretty trivial.

$$\frac{du}{dr} = 4\pi r^2 \rho(p)$$
$$\frac{dp}{dr} = -\frac{p+\rho(p)}{r(r-2u)}\left(4\pi p r^3 + u\right)$$

$p = -\rho$ The change in the pressure vanishes.

ρ is constant. You will do this one.

Another Ingredient

- The equation of state is a relationship between the pressure and density of a material.
- Some examples:

$$p = \frac{1}{\mu m_u} \rho k T \qquad \text{classical ideal gas}$$

$$p = A \rho^{c_p / c_v} \quad \text{isentropic equation of state}$$

- Neutron stars are effectively cold so these equations of state don’t cut it.

Degeneracy Pressure (1)

- The combined wavefunction of a bunch of fermions (like electrons and neutrons) must be anti-symmetric. For example,

$$\Psi(A,B) = \frac{1}{\sqrt{2}}(\psi_1(A)\psi_2(B) - \psi_1(B)\psi_2(A))$$

- What happens if state "1" and "2" are the same state?

$$\Psi(A,B) = \frac{1}{\sqrt{2}}(\psi_1(A)\psi_1(B) - \psi_1(B)\psi_1(A))$$
$$= 0$$

Degeneracy Pressure (2)

- So each fermion must be in a different state.
- If there are no forces, the spatial eigenfunctions are simply plane waves with a particular momentum ***p***.
- If the temperature vanishes, the fermions fill each available momentum state up to a certain energy (E_F - the Fermi energy).

Degeneracy Pressure (3)

- Let's count up the states up to the Fermi energy. First let's define the Fermi momentum: $E_F^2 = p_F^2 c^2 + m^2 c^4.$
- Integrate over momentum space:
$$n = \int \frac{d^3 \mathbb{N}}{d^3x\, d^3p} d^3p = \frac{g}{h^3} \int_0^{p_F} 4\pi p^2 dp = \frac{8\pi}{3h^3} p_F^3$$
- *g* is the multiplicity of a momentum state: *g*=2 for electrons and neutrons, *g*=6 for quarks

Degeneracy Pressure (4)

- The pressure and energy density are given by similar integrals:

$$p = \frac{1}{3}\int pv \frac{d^3\mathbb{N}}{d^3x\,d^3p} d^3p = \frac{g}{h^3}\int_0^{p_F} \frac{p^2c^2}{E} 4\pi p^2 dp$$

$$\rho = \int E \frac{d^3\mathbb{N}}{d^3x\,d^3p} d^3p = \frac{g}{h^3}\int_0^{p_F} E 4\pi p^2 dp$$

Degeneracy Pressure (5)

- If we make the standard definition, $x = \frac{p_F}{mc}$

$$n = \frac{g}{6\pi^2\lambda^3}x^3 \quad \text{where} \quad \lambda = \frac{\hbar}{mc}$$

$$p = \frac{gmc^2}{2\lambda^3}\frac{1}{8\pi^2}\left\{x(1+x^2)^{1/2}(2x^2/3-1)+\ln\left[x+(1+x^2)^{1/2}\right]\right\}$$

$$\rho = \frac{gmc^2}{2\lambda^3}\frac{1}{8\pi^2}\left\{x(1+x^2)^{1/2}(1+2x^2)-\ln\left[x+(1+x^2)^{1/2}\right]\right\}$$

- We have the following limits,

$$p \propto \begin{cases} x^5 & \text{if } x \ll 1 \\ x^4 & \text{if } x \gg 1 \end{cases} \qquad \rho \propto \begin{cases} x^3 & \text{if } x \ll 1 \\ x^4 & \text{if } x \gg 1 \end{cases}$$

Degeneracy Pressure (6)

- In astrophysics, we have two important regimes:
  - Electron supply the pressure and nuclei supply the mass:

$$p \propto \begin{cases} \rho^{5/3} & \text{NR} \\ \rho^{4/3} & \text{UR} \end{cases}$$

  - Neutrons supply the pressure and the mass

$$p \propto \begin{cases} \rho^{5/3} & \text{Non-Relativistic} \\ \rho & \text{Ultra-Relativistic} \end{cases}$$

Fermi Gases in Equilibrium

- In general there are several species in chemical equilibrium: nuclei, neutrons, proton, electrons (and other leptons). An example:

$$n \rightleftharpoons p + e^- + \bar{\nu}_e$$

$$\mu_n = \mu_p + \mu_e + \mu_{\bar{\nu}_e}$$

- For a non-degenerate species μ is essentially the mass of the particle, so if $\mu_e > m_n - m_p$ each new electron added to the gas combines with a proton to make a neutron until nearly all the protons are exhausted.
- For a degenerate species μ is the Fermi energy.

A basic neutron star core (1)

- In the core of a neutron star, you have neutrons, protons and electrons in equilibrium, so $\mu_n = \mu_p + \mu_e$

$$m_n(1 + x_n^2)^{1/2} = m_e(1 + x_e^2)^{1/2} + m_p(1 + x_p^2)^{1/2}$$

- There is also charge balance, $\frac{1}{3\pi^2\lambda_e^3}x_e^3 = \frac{1}{3\pi^2\lambda_p^3}x_p^3$. Therefore $m_e x_e = m_p x_p$.

A basic neutron star core (2)

- Let's eliminate the electrons from the eqn
$m_n(1+x_n^2)^{1/2} = (m_e^2 + m_p^2 x_p^2)^{1/2} + m_p(1+x_p^2)^{1/2}$
and solve for the ratio of protons to neutrons,

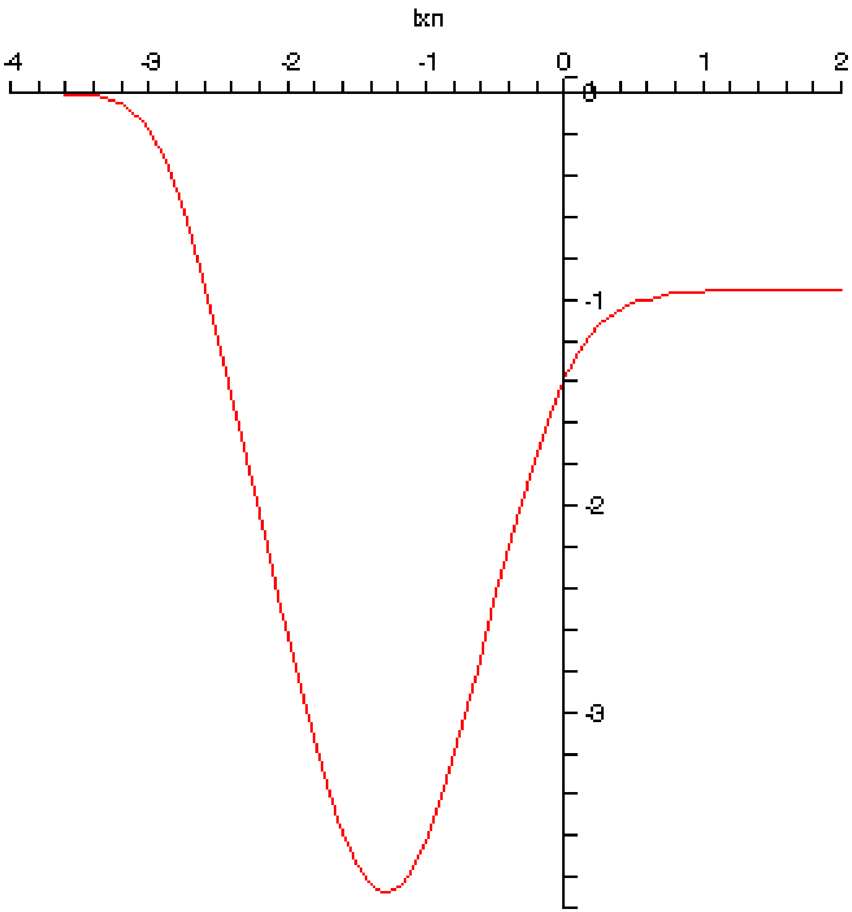
$$\frac{n_p}{n_n} = \left(\frac{m_p x_p}{m_n x_n}\right)^3$$

$$= \frac{1}{8} \frac{1}{(1+x_n^2)^{3/2} x_n^3 m_n^6} \Big\{ [(m_e - m_p)^2 - m_n^2(1+x_n^2)] \times$$

$$[(m_e + m_p)^2 - m_n^2(1+x_n^2)] \Big\}^{3/2}$$

A basic neutron star core (3)

- The neutrons appear at a finite density below which there are only protons and electrons, reach a maximum fraction and asymptote to 8/9 of the baryons.
- Sum over the different particles to get the total pressure.



A basic neutron star crust (1)

- In the crust of a neutron star, you have neutrons, nuclei and electrons in equilibrium. We wish to minimize the total energy density for a given baryon density.

$$\rho = n_N M(A,Z) + \rho_e'(n_e) + \rho_n(n_n)$$

$$\text{where} \;\; \rho_e' = \rho_e - n_e m_e c^2$$

A basic neutron star crust (2)

- The baryon density is *n*, and the total charge vanishes; therefore, it is convenient to define

$$n_e = n(1 - Y_n)\frac{Z}{A} \text{ and } n_n = nY_n$$

$$\rho = n(1 - Y_n)\frac{M(Z,A)}{A} + \rho_e'(n_e) + \rho_n(n_n)$$

- And we take derivatives w.r.t. to A, Z and the densities,

$$\frac{d\rho_e'}{dn_e} = -\frac{\partial M}{\partial Z} = E_{F,e} - m_e c^2 \qquad A^2\frac{\partial}{\partial A}\left(\frac{M}{A}\right) = Z(E_{F,e} - m_e c^2)$$

$$\frac{d\rho_n}{dn_n} = \frac{\partial M}{\partial A} = E_{F,n} \qquad Z\frac{\partial M}{\partial Z} + A\frac{\partial M}{\partial A} - M = 0$$

A basic neutron star crust (3)

- What is *M(Z,A)*? It is the empirically measured atomic weights of various nuclei. Harrison and Wheeler use a fit inspired by the liquid-drop model.

$$
\begin{aligned}
M(Z,A) &= \left[(A-Z)m_n c^2 + Z(m_p+m_e)c^2 - A\bar{E}_b\right] \\
&= m_u c^2 \left[b_1 A + b_2 A^{2/3} - b_3 Z + b_4 A\left(\frac{1}{2}-\frac{Z}{A}\right)^2 + \frac{b_5 Z^2}{A^{1/3}}\right]
\end{aligned}
$$

b_1	0.991749	Nucleon mass - Volume binding energy
b_2	0.01911	Surface tension
b_3	0.000840	Difference between *n* and *p* + *e*
b_4	0.10175	Symmetry energy
b_5	0.000763	Coulomb energy

A basic neutron star crust (4)

- How to calculate the equation of state?
  - Pick a value of *A* and get *Z* from $Z = \left(\frac{b_2}{2b_5}\right)^{1/2} A^{1/2} = 3.54 A^{1/2}$
  - Using *Z* and *A* calculate x_e and x_n from

$$b_3 + b_4\left(1 - \frac{2Z}{A}\right) - 2b_5\frac{Z}{A^{1/3}} = \left[(1 + x_e^2)^{1/2} - 1\right]\frac{m_e}{m_u}$$

$$b_1 + \frac{2b_2 A^{-1/3}}{3} + b_4\left(\frac{1}{4} - \frac{Z^2}{A^2}\right) - \frac{b_5 Z^2}{3A^{4/3}} = (1 + x_n^2)^{1/2}\frac{m_n}{m_u}$$

  - The pressure is the sum of the electron and neutron contributions.
- This is the Harrison-Wheeler Equation of State

A basic neutron star crust (5)

- Key points:
  - There is one type of nucleus at each density starting with ^{56}Fe at low density.
  - Above 10^7 g cm^{-3} iron is no longer.
  - Above 3.2 x 10^{11} g cm^{-3} neutrons drip out of the ^{122}Yt nuclei.
  - Above 4.5 x 10^{12} g cm^{-3} neutrons provide 60% of the pressure and density.
  - N.B. $\rho_{nuc} \sim 10^{15}$ g cm$^{-3.}$

Now for some advanced stuff

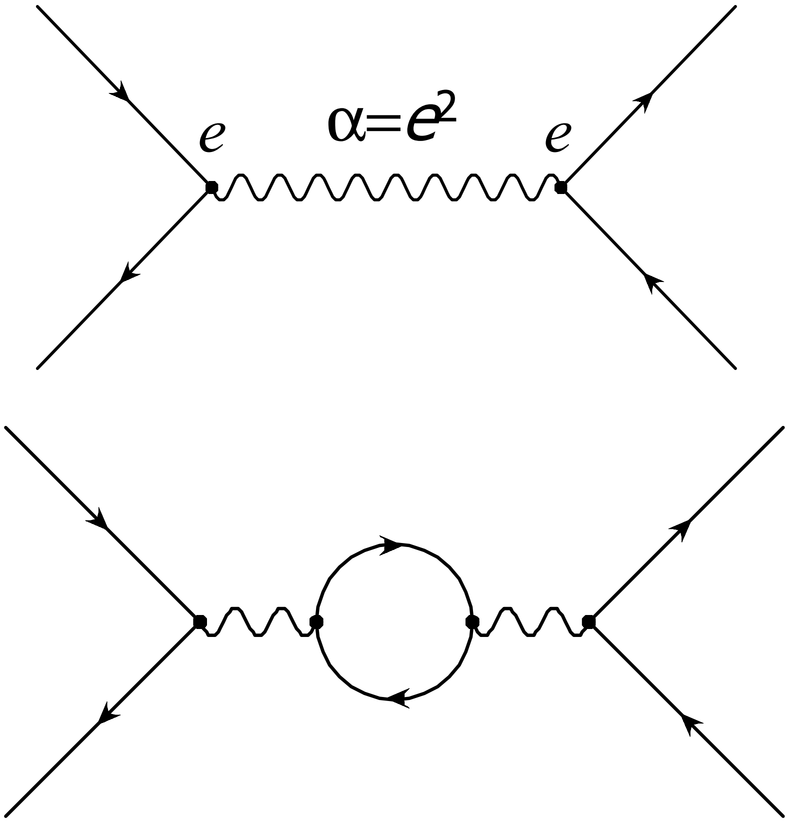
- What is strange quark matter?
  - Massless quarks
  - Massive quarks
- Strange stars and hybrid stars

Nuclear Force

- The quarks in a nucleon are held together by the strong force, a.k.a QCD.
- Why is the strong force “strong”?
  - A force is characterized by a “coupling constant” which quantifies how the particles being pulled together couple to the particles carrying the force.

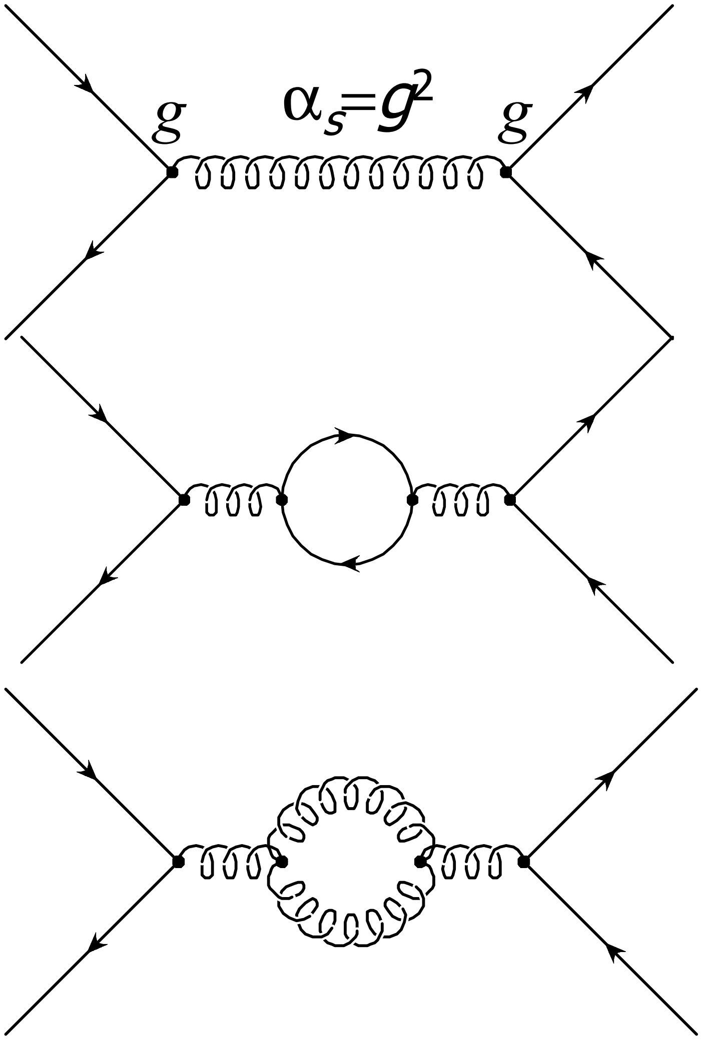
QED

- The classic example is QED.



- Force between two electrons
- Force reduced by vacuum polarization
- α is a function of distance or energy.

QCD is a bit different



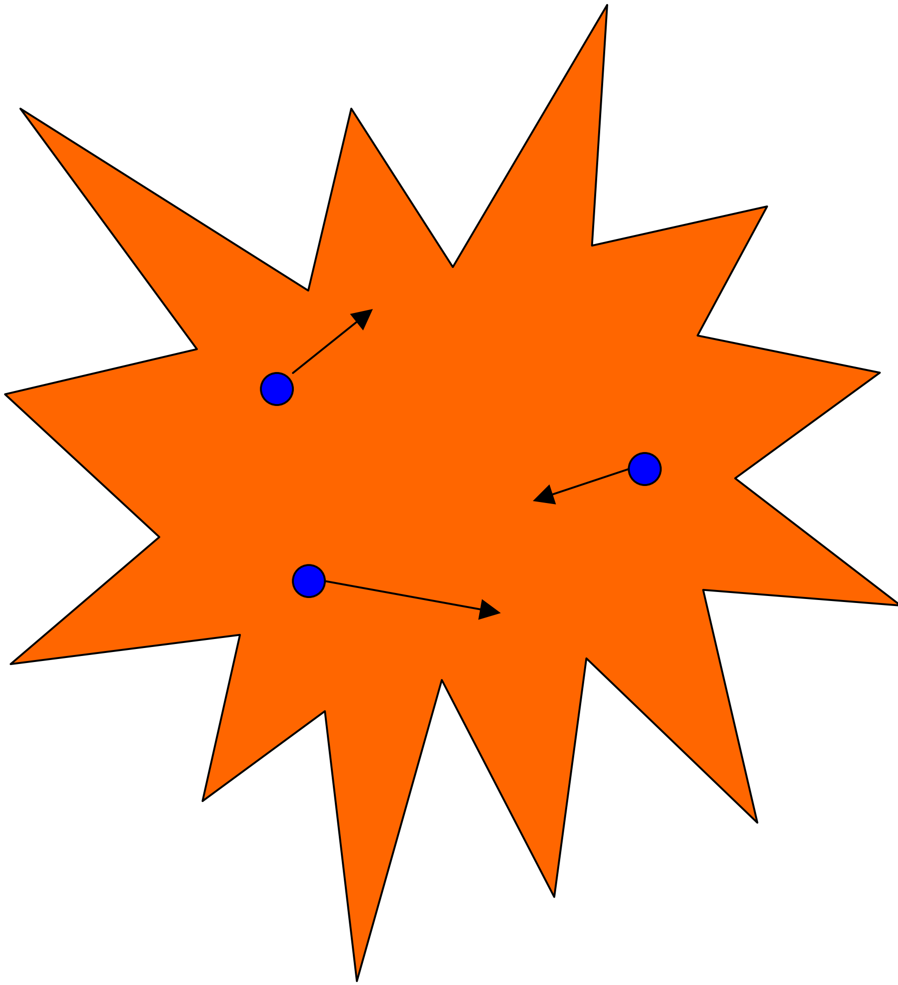
- The lowest order diagram is about the same.
- The vacuum polarization looks similar for the quarks.
- BUT, the gluons couple to other gluons.
- Force increased by polarization.

Asymptotic Freedom

- At low energies, α_s is large. In fact if you try to calculate it perturbatively, it diverges at around 200 MeV (c.f. $\alpha_{QED} \approx 1/137$ at low energies). Quarks are confined.
- However, at higher energies α_s gets smaller (~ 0.15 at 50 GeV). Quarks act free.

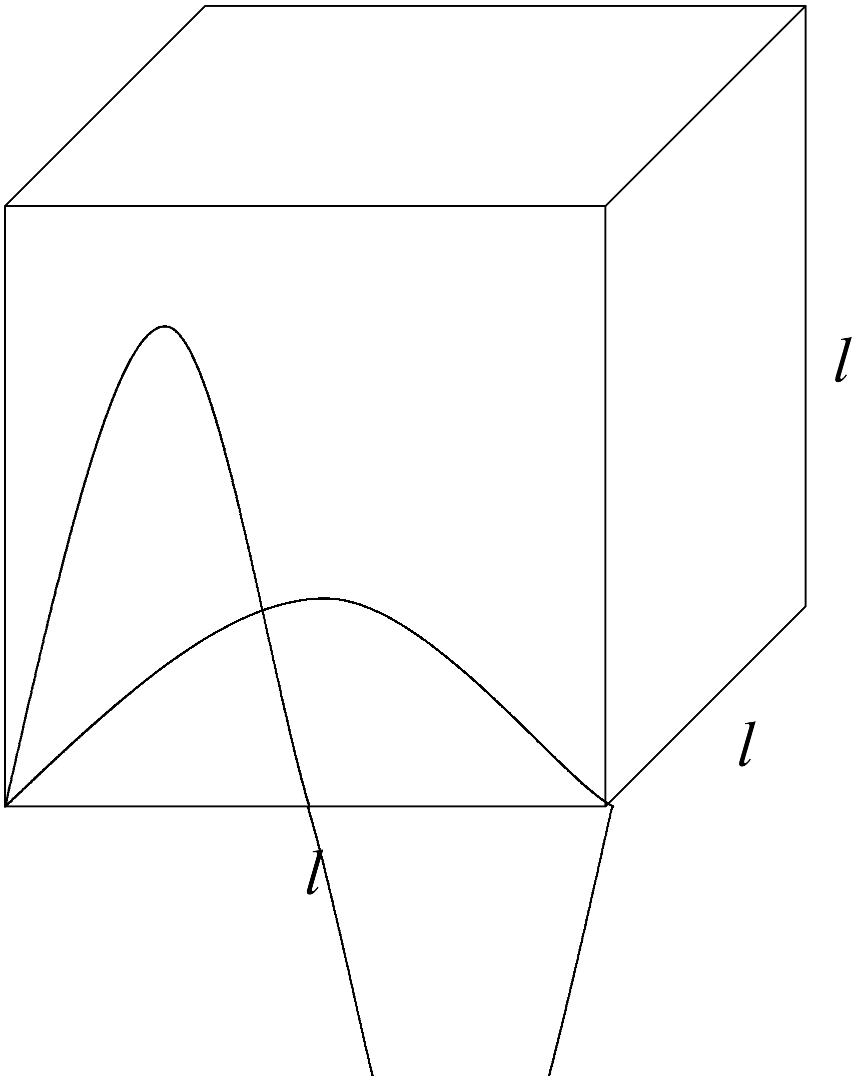
A Heuristic Model

- One way to deal with these is to imagine that quarks are held in a “bag”.
- It costs energy to expand the bag, so the interior of the bag has negative pressure.
- The quarks in the bag act free because they need to have a decent energy to keep the bag from collapsing.



A Cubical Proton

- Let's use this gross model to calculate the mass of a proton and its first excited state.
- Pretend that the quarks are massless and free in the box.
- $E_0 = \frac{hc}{2l}, E_1 = \frac{hc}{l}$



Total Energy

- Ground state: 3 quarks in E_0 plus the bag energy.

$$E_{\text{ground}} = \frac{3}{2}\ \frac{hc}{l} + Bl^3 = 2.4h^{3/4}c^{3/4}B^{1/4}, l^4 = \frac{1\,hc}{2\,B},$$

- Ground state: 2 quarks in $E_{0,}$ 1 quark in E_1 plus the bag energy.

$$E_{\text{excited}} = 2\frac{hc}{l} + Bl^3 = 3.0h^{3/4}c^{3/4}B^{1/4}, l^4 = \frac{1\,hc}{2\,B},$$

How did we do?

- Proton: E=938 MeV, so $B^{1/4} \approx$ 100 MeV in units with $\hbar = c = 1.$

- Our first excited state has E=1171 MeV, compared with 1232 MeV for the Δ resonance.

A Star-Sized Bag

- We have calculated the equation of state for free Fermions already. It is even simpler in the massless limit. To have change neutrality there are equal numbers of up, down and strange quarks.
- They each have the same chemical potential.

$$p = \frac{3}{4\pi^2}\mu^4 - B, \rho = \frac{9}{4\pi^2}\mu^4 + B, n = \frac{\mu^3}{\pi^2}$$

Absolutely Stable Quark Matter

- If the energy per baryon of quark matter at zero pressure is less than that of iron then the quark matter is absolutely stable.

$$0 = \frac{3}{4\pi^2}\mu^4 - B, \rho = 3B + B = 4B,$$

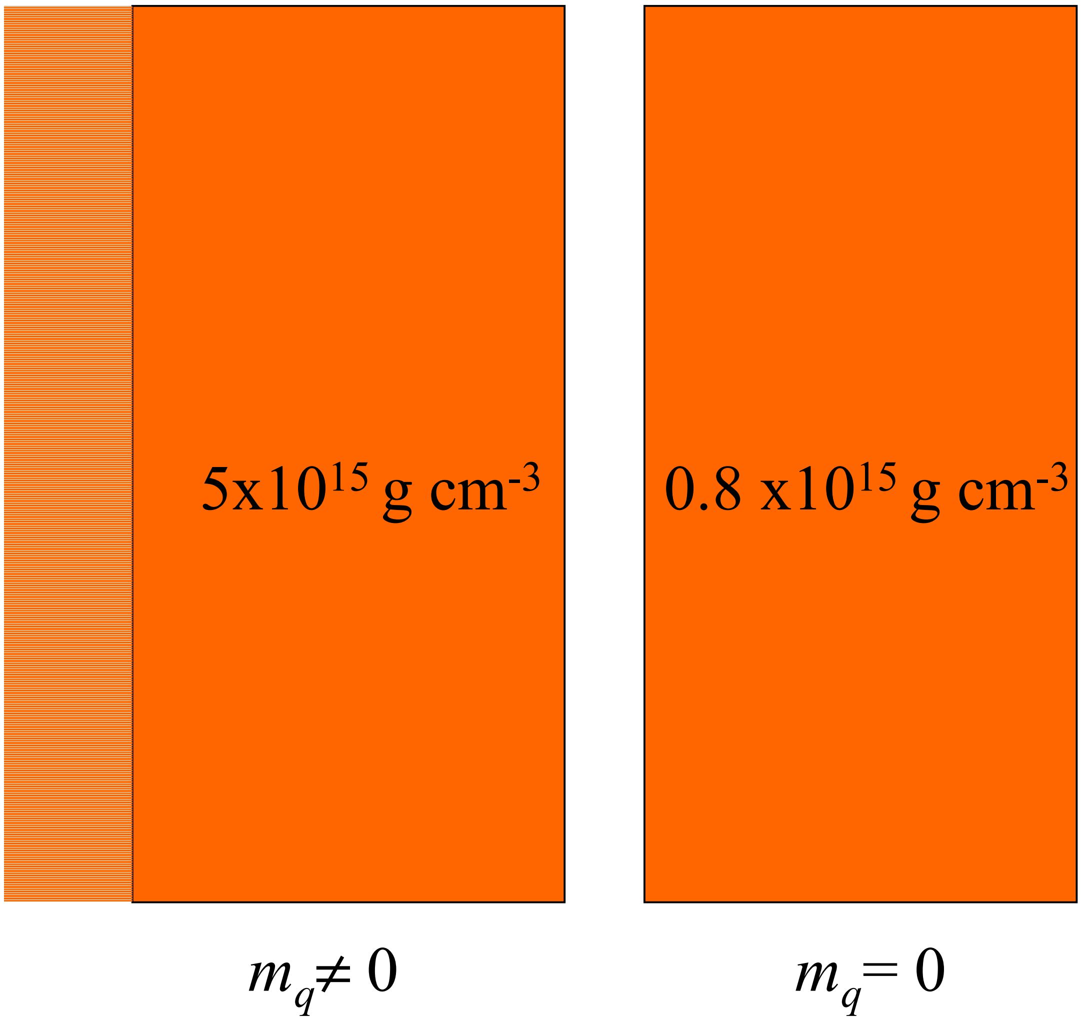
$$n = \frac{\mu^3}{\pi^2}, \quad \text{so} \quad \rho/n = \sqrt{2\pi}\, 3^{3/4} B^{1/4}$$

- Using the value for $B^{1/4}$ we get 570 MeV/b. Fe is 931 MeV/b. The max value of $B^{1/4}$ is 163 MeV.

Massive Quarks

- Quarks actually have masses, 5, 7 and 150 MeV for the up, down and strange respectively.
  - In this case the critical value of $B^{1/4}$ is 155 MeV.
  - Even though the strange quark is much more massive it is present in the mix even at zero pressure.
  - Leptons: electrons and muons also are present (this is important).

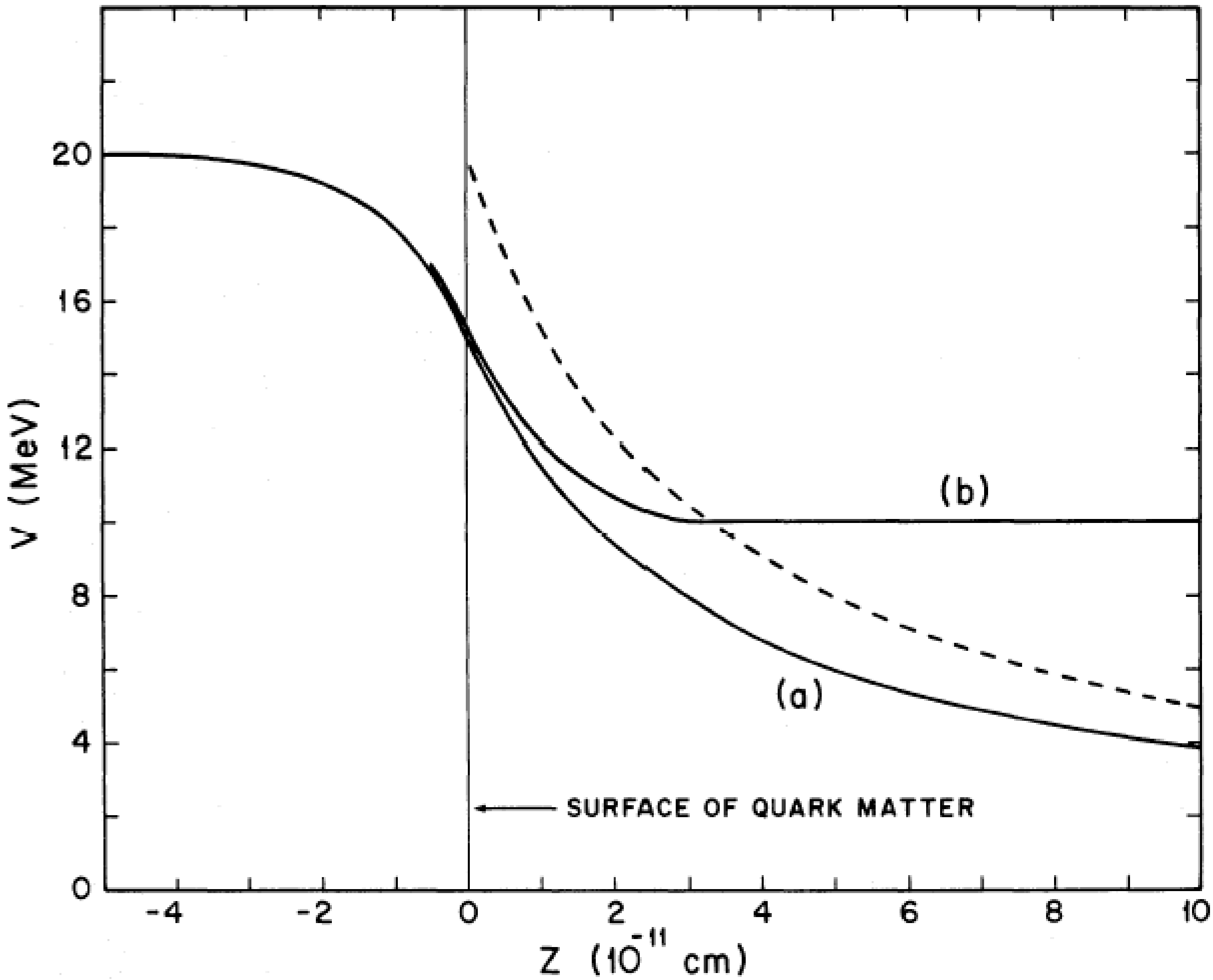
Strange Star Surfaces



- The quarks go from supernuclear to zero density in about 10^{-13}cm, the range of the strong force.
- The leptons don't feel the strong force, so they drop off gradually over a distance ~ λ_e.

Surface Electric Field

- This electric field keeps the SQM from acting like Ice9.
- It also prevents the GJ mechanism from working unless the star accretes a surface layer of normal matter.



Negative Pressure in GR

- Let's replace $\rho \to \rho + B$ and $p \to p - B$ in the OV equations.

$$\frac{du}{dr} = 4\pi r^2(\rho(p) + B) \ \text{ so } \ u' = u + \frac{4\pi}{3} r^3 B$$

$$\frac{dp}{dr} = -\frac{p + \rho(p)}{r(r - 2u')}\left(4\pi(p - B)r^3 + u + \frac{4\pi}{3} r^3 B\right)$$

$$= -\frac{p + \rho(p)}{r(r - 2u')}\left(4\pi p r^3 + u - \frac{8\pi}{3} r^3 B\right)$$

- The negative pressure reduces the pressure gradient until $u' \sim r/2$.

Neutron and Quark Stars