# Rotation-Powered Neutron Stars

Spinning magnets in the sky

# Are pulsars rotating neutron stars?

- Things to remember:
  - Periods range from 1.6 ms to 8 s.
  - Pulsar periods increase very slowly and don’t decrease except for glitches.
  - Pulsars are stable clocks.
- Size: $r < cP < 500$ km so it could be a white dwarf, black hole or neutron star.

# Maximal Rotation Frequencies

- Equate the centripetal force to the gravitational force at the surface:

$$\Omega^2 R < \frac{GM}{R^2} \text{ so } \Omega < \left(\frac{GM}{R^3}\right)^{1/2}$$

- Using $\rho \sim 10^8$ g cm$^{-3}$ gives $\Omega \sim 5.3$ Hz or $P \sim 1$ s (a white dwarf can't spin that fast)
- Using $\rho \sim 10^{15}$ g cm$^{-3}$ gives $\Omega \sim 16$ kHz or $P \sim$ 0.4ms (a neutron star can spin fast enough)

# Pulsation Frequencies

- The fast pulsation modes of a star are pressure modes, i.e. sound waves.
  - We need to estimate the speed of sound

$$c_s^2 = \frac{dP}{d\rho} \sim \frac{P}{\rho}$$

  - We have an estimate for the density but what about *P*? For a constant density star, the gravitational acceleration is proportional to the distance from the center!

$$P = \int_0^R \frac{GM}{R^2}\rho\frac{r}{R}dr = \frac{GM}{R^2}\rho\frac{R}{2}$$

$$c_s^2 \sim \frac{P}{\rho} = \frac{GM}{2R} \quad \omega = \frac{2\pi c_s}{R} = 2\pi\left(\frac{GM}{2R^3}\right)^{1/2}$$

# Neutron Stars and Black Holes

- Both the maximal rotation frequency and the typical pulsation frequency of white dwarfs fall short so we are left with neutron stars and black holes.
- Isolated black holes have no structure to emit periodically and material in orbit around a BH would spiral in and the period would decrease.
- Ditto for neutron star binaries
- Pulsation modes of a neutron star fit the bill for the period, BUT the period would typically decrease as the energy in the mode dissipates.

# The Big Flywheel

- If a neutron star is born spinning near break-up, it has as much rotational energy as a supernova.
- If there only was a way to convert that energy into radio waves.
- Hmmmm……

# Magnetic Dipole Radiation (1)

- Regardless of what's going on inside of the star, the magnetic dipole moment is

$$|\mathbf{m}| = \frac{\mathbf{B_p R^3}}{\mathbf{2}}$$

where $B_p$ is the strength of the dipole field at the pole.

- If the dipole moment varies with time, energy is radiated at a rate of

$$\dot{E} = -\frac{2}{3c^3}|\ddot{\mathbf{m}}|^2$$

- Suppose that the magnetic axis is not aligned with the rotation axis ($\alpha$ is the angle between the axes).

$$\mathbf{m} = |\mathbf{m}| \begin{bmatrix} \cos\alpha \\ \sin\alpha\cos\phi \\ \sin\alpha\sin\phi \end{bmatrix}$$

$$\phi = \Omega t$$

$$|\ddot{m}| = \Omega^2 \sin\alpha |\mathbf{m}|$$

$$\dot{E} = -\frac{B_p^2 R^6 \Omega^4 \sin^2\alpha}{6c^3}$$

# Magnetic Dipole Radiation (2)

- The total rotational energy of the star is

$$E = \frac{1}{2} I \Omega^2, \dot{E} = I \Omega \dot{\Omega}$$

- Putting things together

$$\dot{\Omega} = -\frac{B_p^2 R^6 \Omega^3 \sin^2 \alpha}{6 c^3 I}$$

- Let's define a characteristic time,

$$T = -\frac{\Omega_0}{\dot{\Omega}_0} = \frac{6 c^3 I}{B_p^2 R^6 \Omega_0^2 \sin^2 \alpha}$$

- This gives us

$$\dot{\Omega} = -\frac{\Omega}{T} \left( \frac{\Omega}{\Omega_0} \right)^2$$

- Separating and integrating,

$$\frac{1}{\Omega^3} d\Omega = -\frac{1}{T \Omega_0^2} dt$$

$$\frac{1}{2\Omega_0^2} - \frac{1}{2\Omega_i^2} = \frac{t_0 - t_i}{T \Omega_0^2}$$

- Let's assume that at $t_i$, $P$=0,

$$\frac{1}{2\Omega_0^2} = \frac{t_0 - t_i}{T \Omega_0^2}; \quad \tau = t_0 - t_i = \frac{1}{2} T$$

# P and P-dot

- Although theoretically it is natural to talk about the frequency, observationally people talk about the period, $P$=2π/Ω and d$P$/d$t$=-2π/$\Omega^2$ dΩ/d$t$ , a.k.a. P-dot.

$$T = -\frac{\Omega_0}{\dot{\Omega}_0} = \frac{P}{\dot{P}}$$

- If you can estimate $I$ and $R$, you can get an estimate of $B_p$

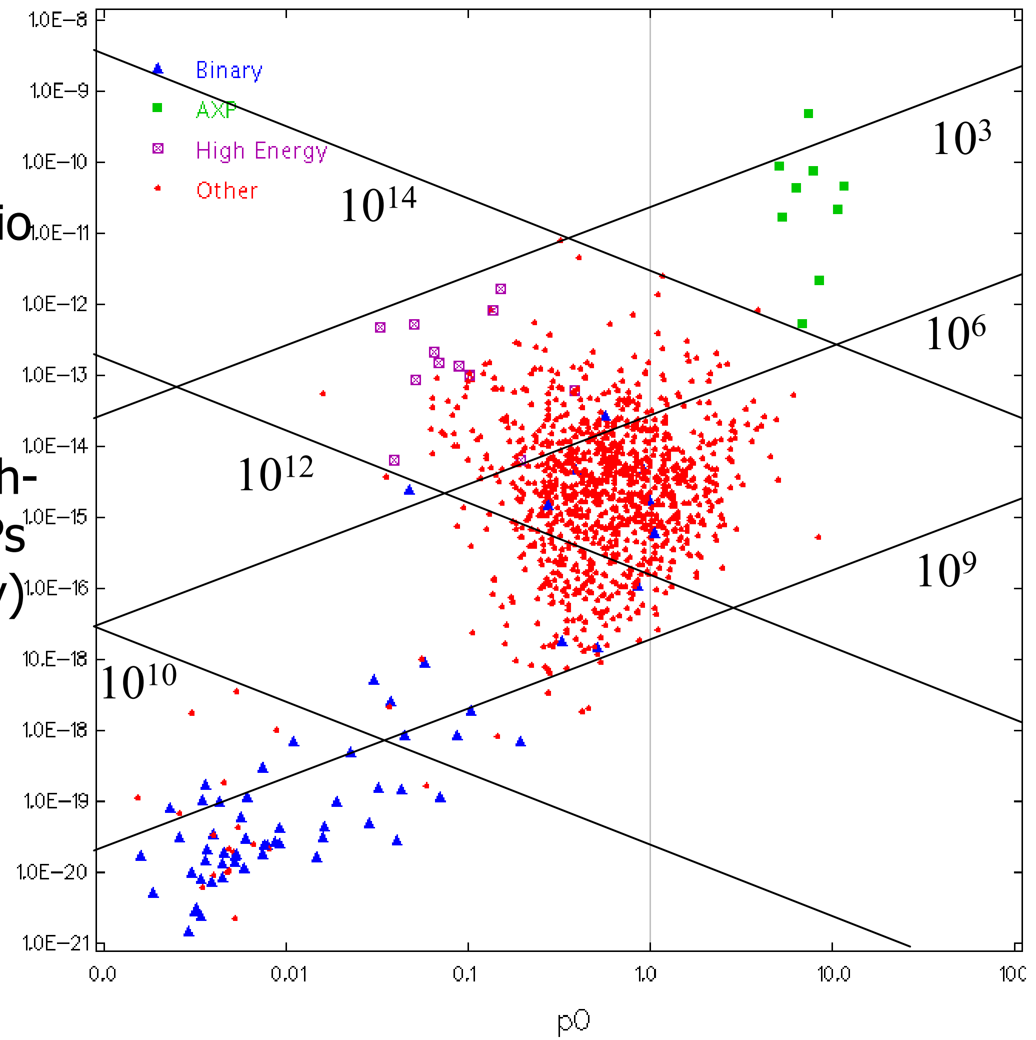
$$B_p^2 \sin^2 \alpha = \frac{6c^3 I P \dot{P}}{4\pi^2 R^6}$$

$$B_p \sin \alpha = 6.4 \times 10^{19} I_{45} R_6^{-6} (P_1 \dot{P})^{1/2} \ \mathrm{G}$$

- Some examples:
- Crab: $P$=0.033s, $P$-dot=4 x $10^{-13}$
  - $B_p$= 7 x $10^{12}$ G, $T$/2=1300 yr
- Vela: $P$=0.089s, $P$-dot=1 x $10^{-13}$
  - $B_p$= 6 x $10^{12}$ G, $T$/2=14000 yr
- 1841: $P$=11.77s, $P$-dot=4 x $10^{-11}$
  - $B_p$= 1 x $10^{15}$ G, $T$/2= 4700 yr
- 1937: $P$=0.0016s, $P$-dot=1 x $10^{-19}$
  - $B_p$= 8 x $10^{8}$ G, $T$/2=2.5 x $10^{8}$ yr

# The P-P-dot Diagram!

- Like the H-R diagram.
- Things to notice:
  - most red dots (isolated radio pulsars): $10^{11\text{-}13}$ G, $10^{5\text{-}8}$ yr
  - most PSRs in binaries have short periods
  - Many young PSRs have high-energy emission or are AXPs (no radio and thermal x-ray)



# Another Model (GW)

- A spinning barbell emits gravitational radiation and slows according to

$$\dot{E} = -\frac{32G}{5\ c^5} I^2 \epsilon^2 \Omega^6$$

- Astronomers like power-law models, so take

$$\dot{\Omega} = -A\Omega^n$$

- How can we determine *n*?
  - *n*=3: MD, *n*=5: GW

- Take the time derivative of both sides,

$$\ddot{\Omega} = -An\Omega^{n-1}\dot{\Omega}$$

$$\Omega\ddot{\Omega} = -nA\Omega^n\dot{\Omega} = n\dot{\Omega}^2$$

$$n = \frac{\Omega\ddot{\Omega}}{\dot{\Omega}^2} \quad \tau = \frac{T}{n-1}$$

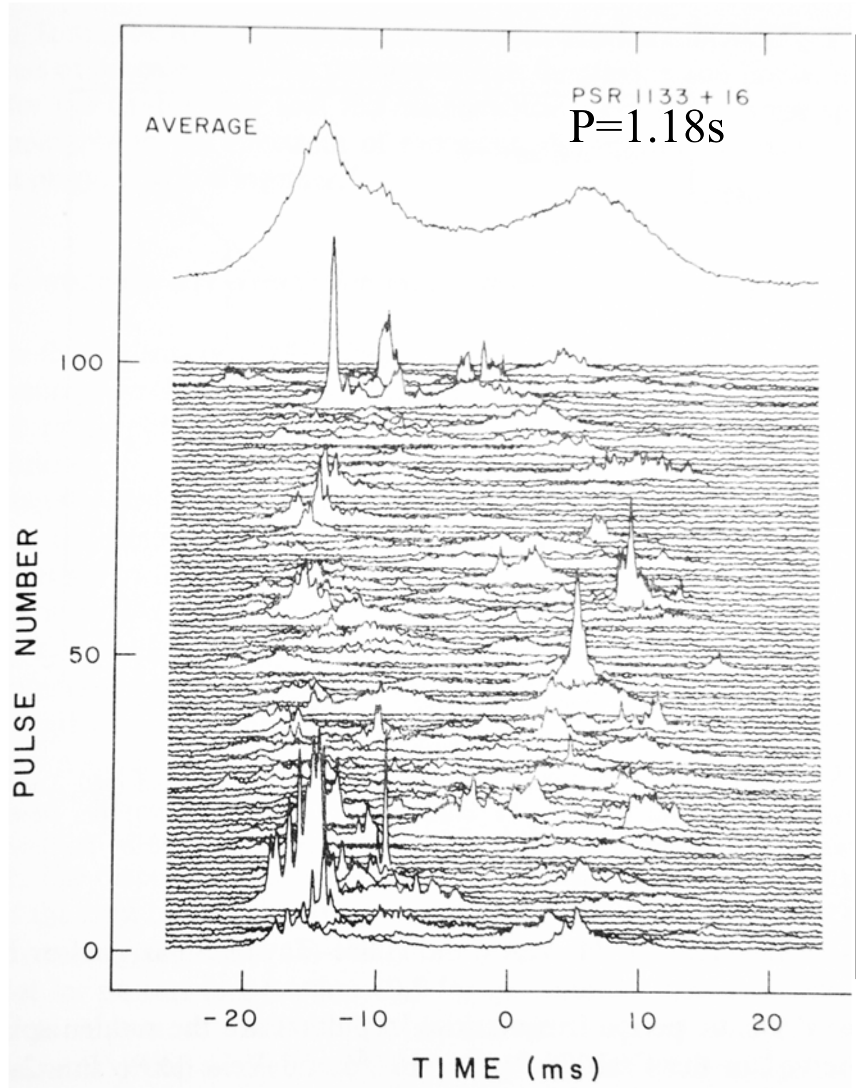
- Unfortunately, *n* is difficult to measure accurately but there is other evidence for the MD model.

# Evidence for Dipole Model

- Measurement of magnetic field strengths from cyclotron lines on Her X-1 gives $4 \times 10^{12}$ G.
- Energy from spin-down of Crab is sufficient to power the Crab nebula.
- Polarization of the radiation is characteristic for a magnetic dipole geometry.

# Pulsar Emission Observed

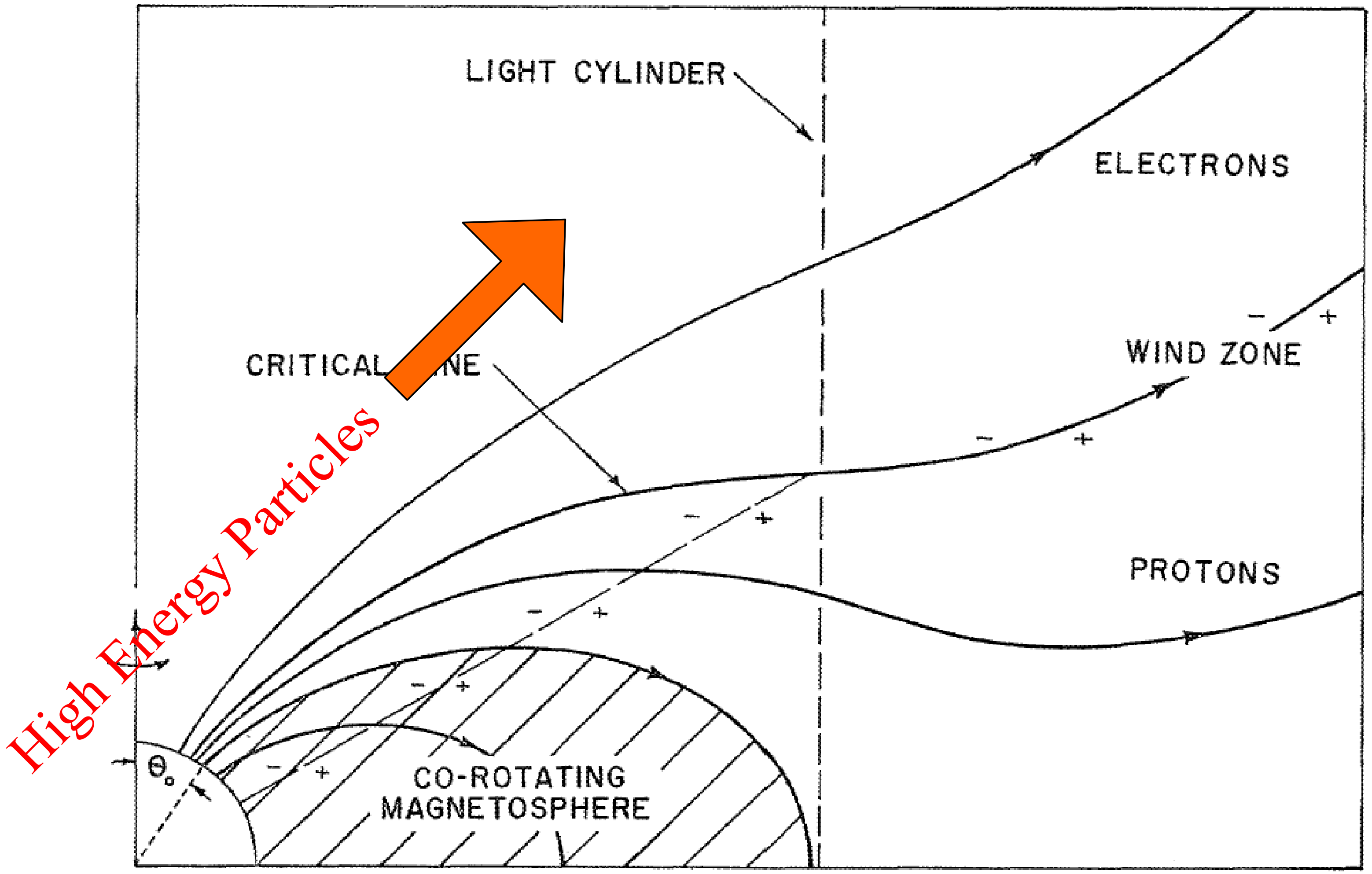
- The individual pulses are quite random.
- The sum of many pulses is constant for a particular pulsar.
- The emitting elements are all in a particular region but not all are active at the same time.



# Pulsar Emission Model

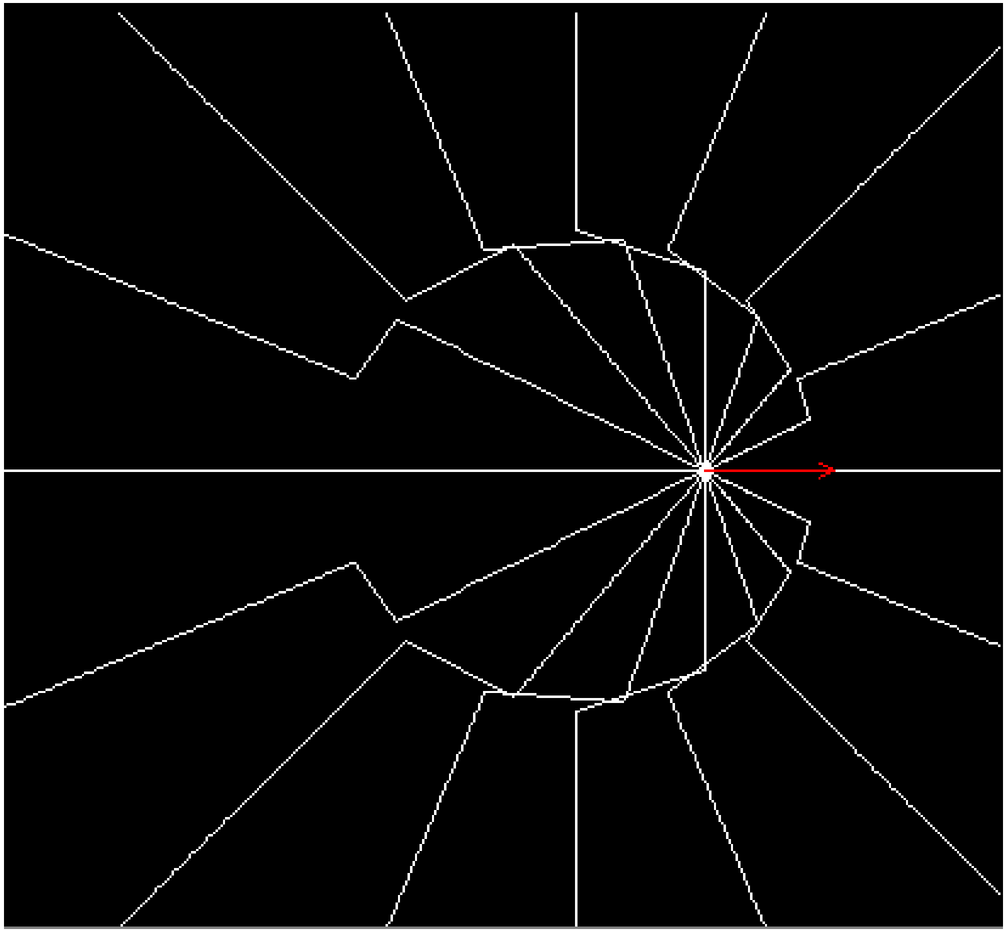
- We understand why pulsars spin down, but why do they emit radio waves.
  - A rotating magnetic dipole emits radiation at the rotation frequency - 0.1-600Hz.
  - Only a tiny fraction of the spin-down energy needs to end up as pulsed radio emission.
- Let’s start with the Goldreich and Julian picture to build up a heuristic model.

# Goldreich-Julian Picture



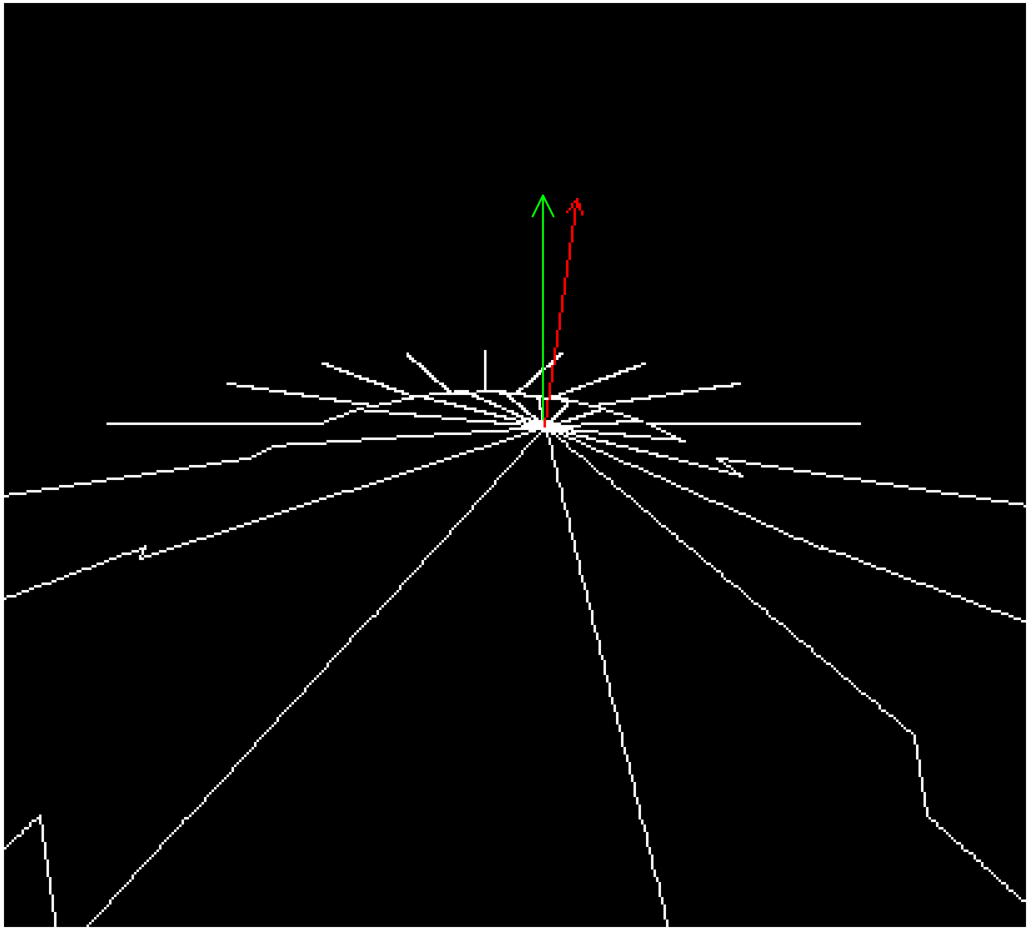
# Curvature Radiation

- Accelerated charges radiate, so the particles travelling along the fields will radiate as the field lines curve.



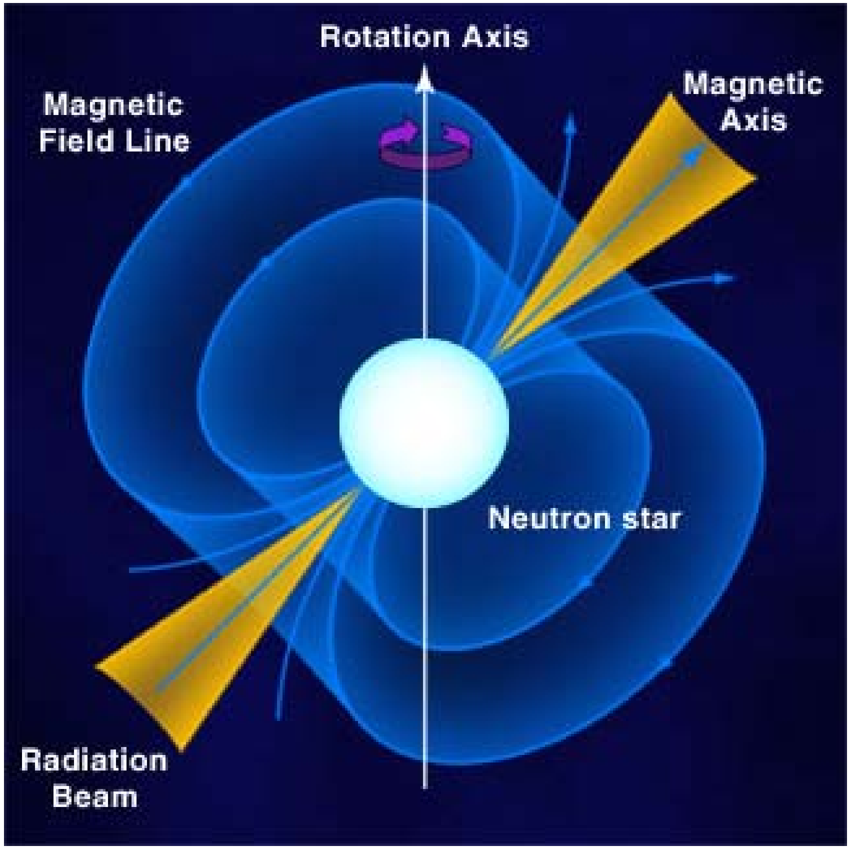
# Let's go relativistic

- The charges travel relativistically.  That makes it even more interesting.
- The radiation is polarized in the direction of the acceleration and it is beamed in the direction of motion!



# We have a model!

- The radiation only comes from where there are high-energy particles - the open-field lines.
- The most intense radiation comes to us from bunches of particles moving toward us relativistically.
- The radiation is polarized along the direction of curvature of the magnetic field lines.



# The Open Field Lines (1)

- Because the radiation only comes from the open field lines, the pulsar can only be seen from within a cone centered on the magnetic pole. This cone sweeps around the sky like a lighthouse beam.
- Let's find the first field line that reaches the light cylinder.
- The equation for a flow/field line is

$$\frac{d\mathbf{x}(\lambda)}{d\lambda} = \mathbf{B}(\mathbf{x}(\lambda)) \quad \Rightarrow \quad \frac{r d\theta}{H_\theta} = \frac{dr}{H_r}$$

# The Open Field Lines (2)

- Filling in the results for a dipole field

$$\frac{dr}{d\theta} = 2r\frac{\cos\,\theta}{\sin\,\theta} \quad \Rightarrow \quad \ln r'\Big|_{r_0}^{r} = 2\ln\sin\theta'\Big|_{\theta_0}^{\theta}$$

$$\frac{r}{r_0} = \frac{\sin^2\theta}{\sin^2\theta_0} \;\text{ and }\; r_{max} = \frac{r_0}{\sin^2\theta_0} \;\text{ for }\; \theta = \frac{\pi}{2}$$

- The radius of the light cylinder is equal to $r_{max}$ for the last closed field line.

$$R_{lc} = \frac{cP}{2\pi} = r_{max} = \frac{r_0}{\sin^2\theta_0} \quad \Rightarrow \quad \sin^2\theta_0 = \frac{2\pi r_0}{cP}$$

$$\sin\theta_0 = 0.014 r_{0,6}^{1/2} P^{-1/2}$$

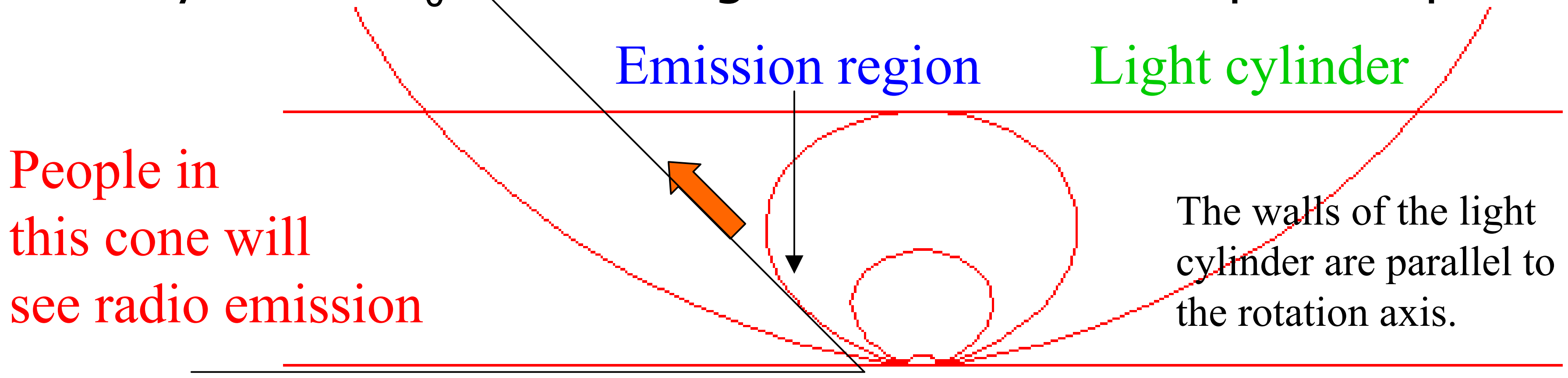
$$\theta_0 = 0.82^\circ r_{0,6}^{1/2} P^{-1/2} \;\text{ for }\; \theta_0 \ll 1$$

# How did we do?

- Empirically they find that the maximum opening angle of the emission is (our line of sight might not cut through the entire polar region)

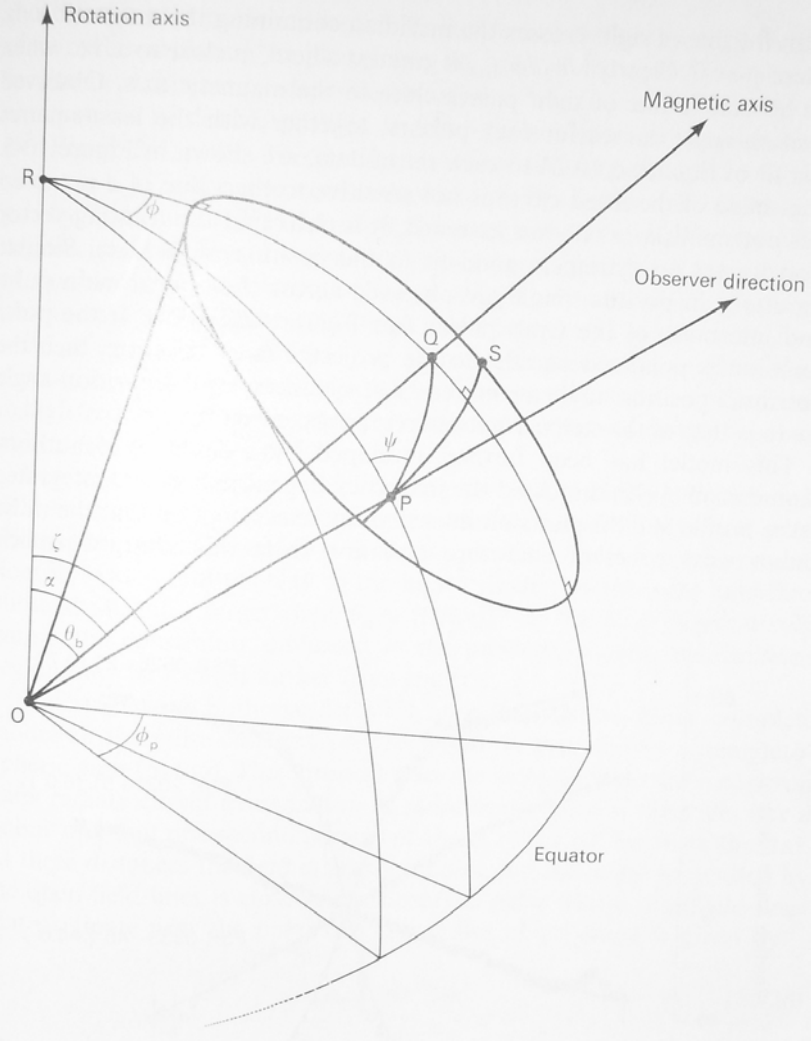
$$\Delta\theta = 5^\circ P^{-\alpha} \text{ where } \alpha = 1/3 - 1/2$$

- So $r_{0,6} \sim 40$ for the emission region and it may be a function of the period.
- If you take $r_0 = R$ then θ gives the size of the polar cap.



# Polarization

- In our model the polarization of the radiation is in the direction that the particles are accelerated.
- This acceleration is always directly away from the dipole axis.



# Break out the spherical trig.

SAP

222 $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

131 $$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

311 $$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

221 $$\cos a \cos C = \sin a \cot b - \sin C \cot B$$

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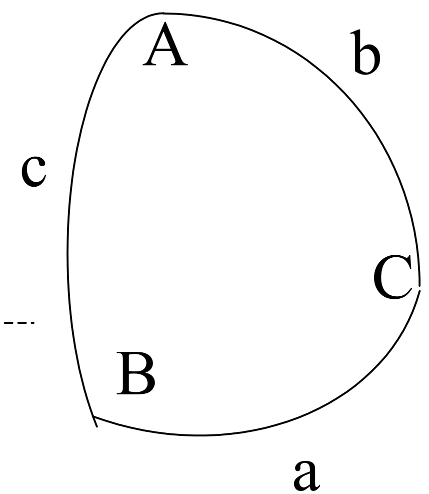
322 $$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$$

311 $$\cos A = \csc b \csc c (\cos a - \cos b \cos c)$$



# For our triangle

- To know which formula to use, you have to know what you have and want.   We have/want two angles (Φ, ψ) and two sides (α,ζ) and only one pair (ψ, α ).

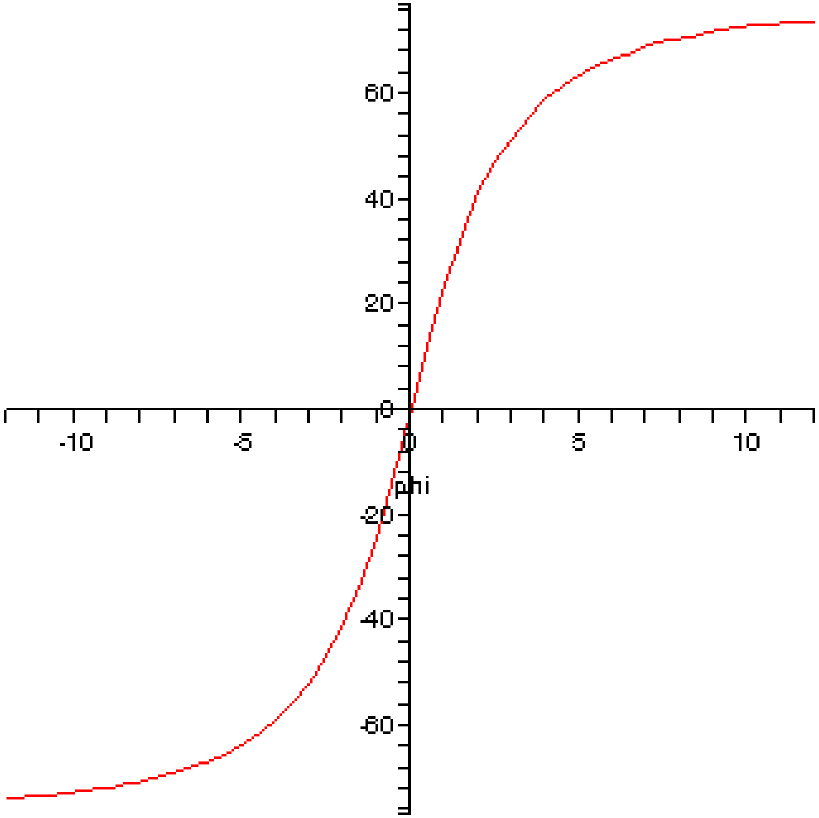
$$\cos a \cos C = \sin a \cot b - \sin C \cot B$$

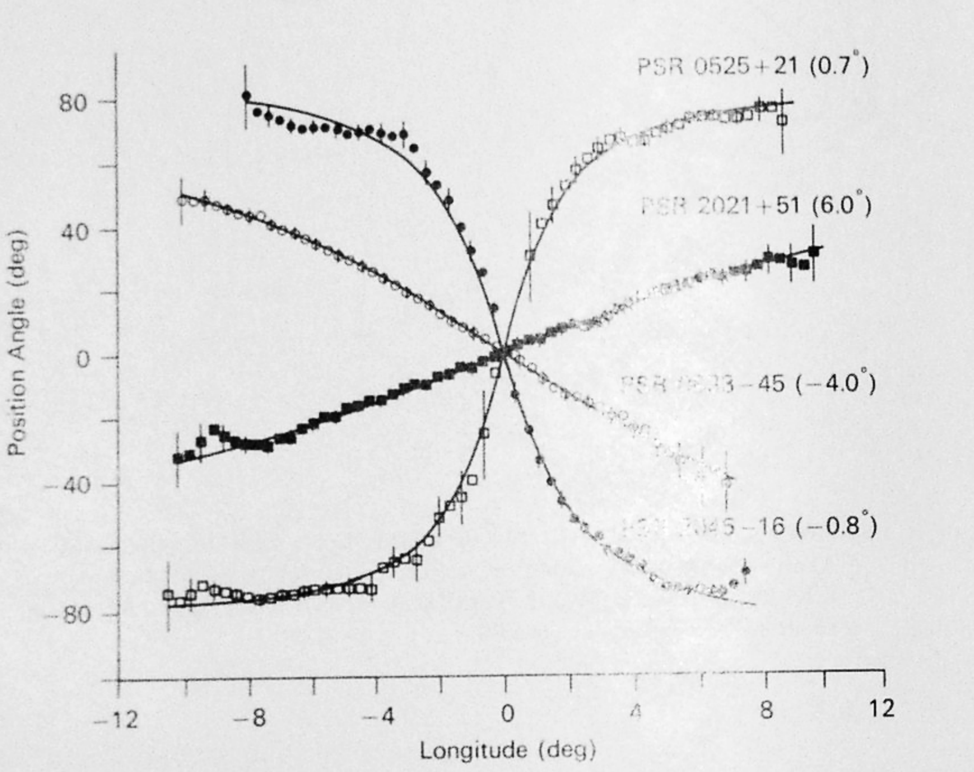
$$\cos \zeta \cos \phi = \sin \zeta \cot \alpha - \sin \phi \cot \psi$$

$$\sin \phi \cot \psi = \sin \zeta \cot \alpha - \cos \zeta \cos \phi$$

$$\tan \psi = \frac{\sin \phi \sin \alpha}{\sin \zeta \cos \alpha - \cos \zeta \sin \alpha \cos \phi}$$

# Theory and Observations



# What are pulsars good for?

- Probing the properties of our Galaxy
  - The Dispersion Measure and Rotation Measure
- Probing the properties of spacetime
  - Gravitational radiation from binary neutron stars

# Dispersion Measure Redux

- If you remember from last week, the arrival time of pulses depends on the frequency:

$$t_2 - t_1 = \frac{2\pi e^2}{mc}(\omega_2^{-2} - \omega_1^{-2}) \int_0^d n_e dl$$

- The dispersion constant is

$$D = (t_2 - t_1)/(\nu_2^{-2} - \nu_1^{-2})$$

and the dispersion measure is

$$DM \ (\mathrm{cm}^{-3}\mathrm{pc}) = 2.410 \times 10^{-16} D \ (\mathrm{Hz})$$

$$DM = \int_0^d n_e dl$$

# More on polarization

- The observed polarization of pulsar radiation depends on frequency.
- What we derived earlier said that the polarization is in the direction that the particles are accelerated (**period**). There was no frequency dependence.
- What is up?

# Magnetized Plasmas

- In a plasma there are two important frequencies: the plasma frequency and the cyclotron frequency.

$$\omega_p^2 = \frac{4\pi n_e e^2}{m_e} \text{ and } \omega_c = \frac{eB}{m_e c}$$

- We already know about the first one -- a passing EM wave induces currents in the plasma.

# An electron in a magnetic field

- If we have an electron in a magnetic field, the force is
  $$\frac{d}{dt}(\gamma m_e \mathbf{v}) = \mathbf{F} = e\frac{\mathbf{v}}{c} \times \mathbf{B} \quad \text{so} \quad \dot{\mathbf{v}} \perp \mathbf{B} \quad \text{and} \quad \dot{\mathbf{v}} \perp \mathbf{v}$$
- We find that
  $$\dot{v}_{\|} = 0 \quad \text{and} \quad \dot{\mathbf{v}}_{\perp} = \frac{q}{\gamma m c}\mathbf{v}_{\perp} \times \mathbf{B}$$
  so we have uniform circular motion around the field line with $\omega_g = \frac{eB}{\gamma m_e c}$. For non-relativistic electrons γ=1.

# A photon runs through it.

- Photons with $\omega < \omega_p$ are absorbed.
- If $\omega > \omega_p$ the photons can propagate.
  - For $\omega < \omega_c$ the photons cannot excite motion across the field, so photons with **e**||**B** travel slower that photons with **e**⊥**B**.
  - For $\omega > \omega_c$ the photons which excite the electrons to spiral the right way are more strongly coupled: one circular polarization travels slower than the other.

# Faraday rotation

- For $\omega > \omega_c$ the plane of polarization of a linearly polarized wave rotates as it propagates through the plasma.

$$\Delta\psi = \frac{2\pi e^3}{m^2 c^2 \omega^2} \int_0^d n_e B \cos\theta dl$$

$$\Delta\psi = RM\lambda^2 \quad \text{where}$$

$$RM = \frac{e^3}{2\pi m^2 c^4} \int_0^d n_e B \cos\theta dl$$

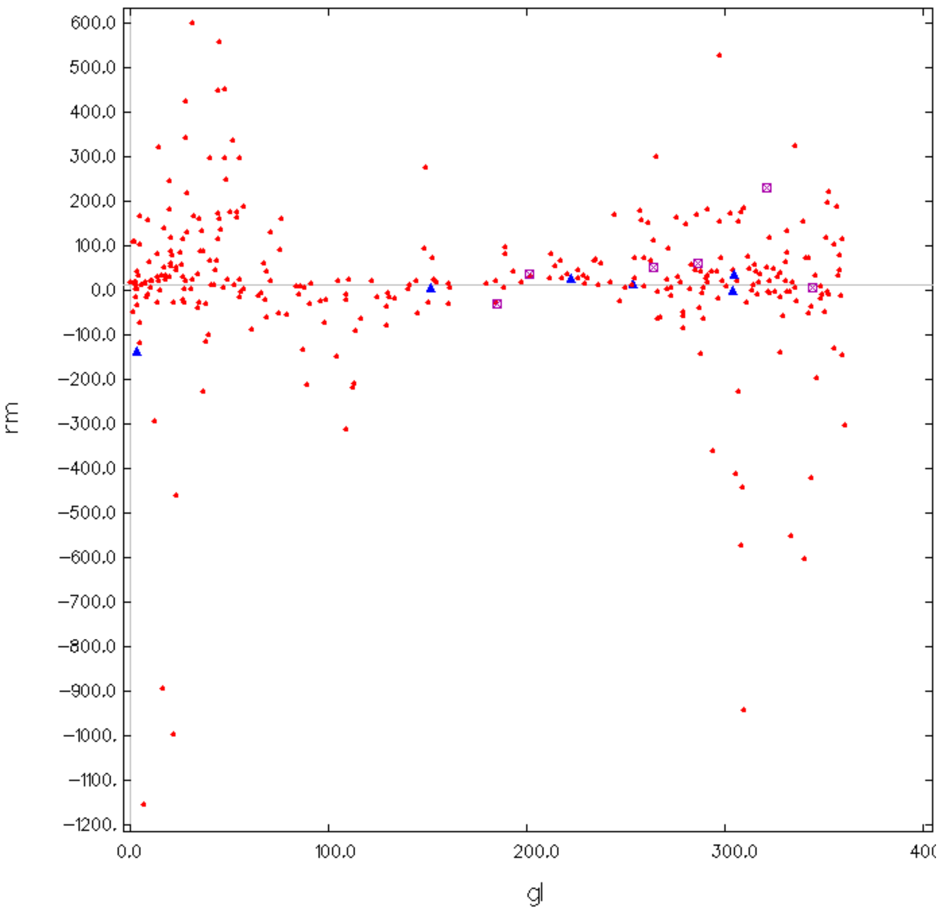
# RM and DM

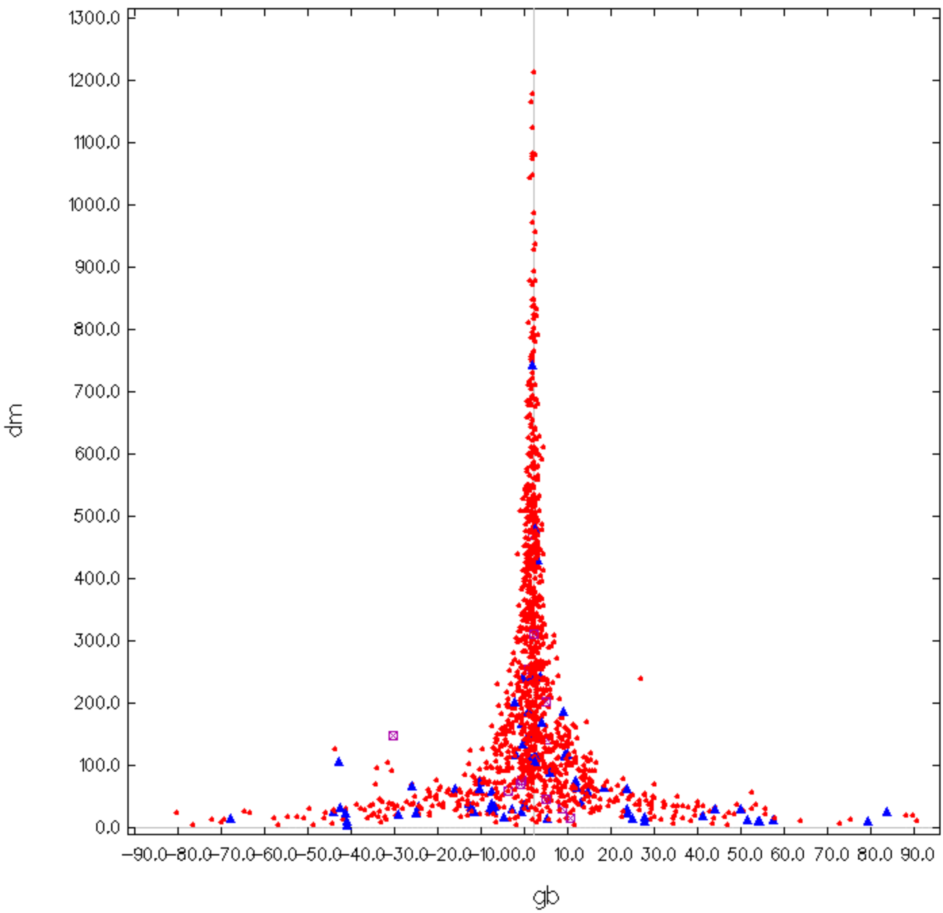
- If we combine the RM and DM for a particular pulsar we get,

$$< B\cos\theta >= \frac{\int_0^d n_e B \cos\,\theta dl}{\int_0^d n_e dl} = \frac{1.232RM}{DM}$$

where *B* is in μG, *RM* is in rad $m^{-2}$ and DM is in $cm^{-3}$ pc.

# Probing Galactic Structure



# Gravitational Radiation

- We have seen gravitational radiation in two contexts so far:
  - The orbital evolution of LMXBs
  - The spin evolution of neutron stars
- We are going to calculate the evolution of a circular orbit explicitly using the quadrupole radiation formula.

# Orbiting neutron stars

- The quadrupole formula gives

$$\dot{E} = -\frac{32G}{5\ c^5} I^2 \epsilon^2 \Omega^6$$

- $I\varepsilon$ is the difference between the moment of inertia along the orbital separation and across it.

# A Diagram

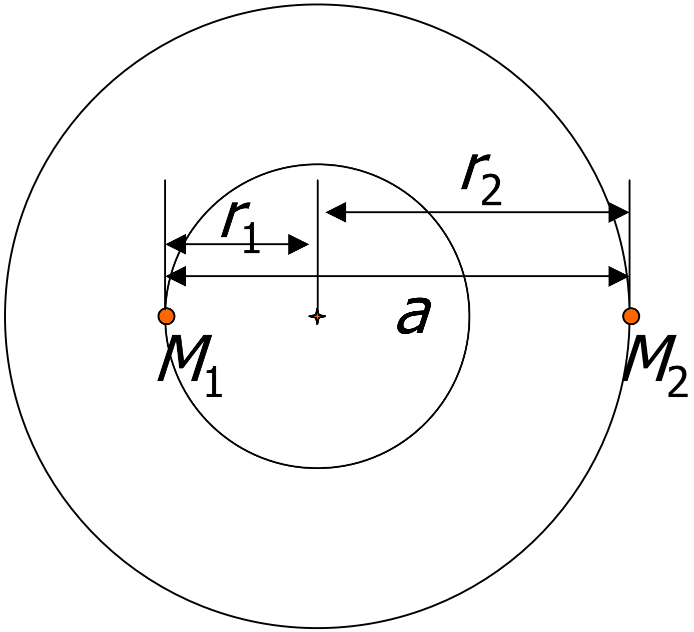
- The two stars orbit about their mutual center of mass.

  $r_1 = a\, M_2 / M$

  and $r_2 = a\, M_1 / M$, so
- $I = M_1 r_1^2 + M_2 r_2^2$

  $= \mu a^2$

  and ε=1.



# Orbital Energy

- The gravitational radiation comes from the energy of the orbit.

$$E = -\frac{GM_1M_2}{a} + \frac{1}{2}I\Omega^2 \text{ with } \Omega^2 a^3 = GM$$

$$= -\frac{GM_1M_2}{a} + \frac{1}{2}I\frac{GM}{a^3}$$

$$= -\frac{GM_1M_2}{a} + \frac{1}{2}\mu a^2\frac{GM}{a^3}$$

$$= -\frac{GM_1M_2}{a} + \frac{1}{2}\frac{GM_1M_2}{a} = -\frac{1}{2}I\Omega^2$$

# Orbital Evolution (1)

- We would like to eliminate *a* in favor of $\Omega$, using Kepler's third law:

$$I = \mu a^2 = \mu \left(\frac{GM}{\Omega^2}\right)^{2/3}$$

- Substituting into the energy equation:

$$E = -\frac{1}{2} I \Omega^2 = -\frac{1}{2} \mu \left(GM\Omega\right)^{2/3}$$

$$\dot{E} = -\frac{1}{3} \mu \left(GM\right)^{2/3} \Omega^{-1/3} \dot{\Omega}$$

# Orbital Evolution (2)

- Putting together the energy equations,

$$-\frac{1}{3}\mu\left(GM\right)^{2/3}\Omega^{-1/3}\dot{\Omega} = -\frac{32G}{5\ c^5}I^2\epsilon^2\Omega^6$$

$$-\frac{1}{3}\mu\left(GM\right)^{2/3}\Omega^{-1/3}\dot{\Omega} = -\frac{32G}{5\ c^5}\mu^2\left(\frac{GM}{\Omega^2}\right)^{4/3}\Omega^6$$

and isolating the change in Ω gives

$$\dot{\Omega} = \frac{96G}{5\ c^5}\mu\left(GM\right)^{2/3}\Omega^{11/3}$$

- What is different about this formula?

# Orbital Evolution (3)

- Doing what we did for the spin,

$$\dot{\Omega} = \frac{\Omega}{T}\left(\frac{\Omega}{\Omega_0}\right)^{8/3}$$

$$\text{where} \quad T = \frac{5}{96}\frac{c^5}{G^{5/3}}\frac{1}{\mu M^{2/3}\Omega_0^{8/3}}$$

$$-\frac{3}{8}\frac{1}{\Omega^{8/3}}\Bigg|_{\Omega_0}^{\Omega_f} = \frac{t_f - t_0}{T\Omega_0^{8/3}} \quad \text{taking} \quad t_0 = 0, \Omega_f \to \infty,$$

$$t_f = \frac{3}{8}T = \frac{5}{256}\frac{c^5}{G^3}\frac{a_0^4}{\mu M^2} \quad \text{and} \quad \Omega = \Omega_0\left(\frac{t_f}{t_f - t}\right)^{3/8}$$

# Sample Evolution

- Let's take two neutron stars with each with a mass of $1.4\mathrm{M}_\odot$, and the wave frequency ( $2f=\Omega/\pi$ ) starting with what we can barely hear (e.g. 30Hz).

| $2f=\Omega/\pi$ | $t_f$ |
|---|---|
| 1Hz | 5.36 days |
| 30 Hz | 53.3 seconds |
| 300 Hz | 115 ms |
| 2175 Hz | 582 μs |

# Some Binary Pulsars

| Name | Orbital Period (hr) | Our $t_f$ (Myr) | Careful $t_f$ (Myr) |
|---|---|---|---|
| B1913+16 | 7.75 | 1600 | 320 |
| B1534+12 | 10.1 | 3400 | 2900 |
| J0757-3039 | 2.4 | 72 | 85 |