Size and mass of many common objects

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A graph of the size and mass of typical objects has a lot of surprising features, both in the details of comparisons of objects and in the overall layout of where the objects fall. Before we get to draw any conclusions, let's take a moment to look at what goes into making such a graph. The first graph of this sort I have seen was made by John Archibald Wheeler.

Making the graph: First, one must decide what ranges of mass and size are needed if the graph will contain a full list of objects. Elelctrons are 10^{-30} kg, and neutrinos are at least one million times lighter than that, so the starting at 10^{-40} kg will let us show most things we might think of. The sun is 10^{30} kg, but our galaxy contains 10^{11} stars, and we want to put in the mass of large groups of galaxies. We really are trying to get *everything* onto one graph, so we better extend up to at least 10^{55} kg. With this huge range of sizes we will have to plot logarithms on both axes. Otherwise everything smaller than a galaxy will be crowded into a few microns at one corner of the graph.

The highest energy cosmic rays detected have a wavelength of 10^{-27}m, but let's extend the graph at lease to 10^{-35}m. The universe is over 10 billion years old, so we might want distances a bit larger than that age times the speed of light. 10^{30}m will do.

One conclusion we can reach already is that with 95 orders of magnitude on the vertical scale and 65 on the horizontal, we will hardly be able to see a factor of two. We will not need to be very precise when we look up weights and sizes.

Plotting objects: Now, we can start looking things up and putting them on the graph. There is a definite bias here towards objects which are well characterized by a size and a density. A person is on the list, my third birthday is not. The solar system is on the list, the path the moon takes in a year is not. This bias is deliberate because we will use the graph to help us understand how objects are put together. Many of the objects we can look up easily.

The great pyramid of Khufu at Giza is 230 m on a side 146 m high. It is made of sandstone (density= 2500 kg/m^3). Therefore it weighs 6.4×10^9kg. I should not have been surprised that a large cloud is *heavier* than a large pyramid, but I was. A 10 km cube of air weighs $7. \times 10^{11}$ kg, even including the fact that the top of the cloud is a lot less dense than the bottom.

To get the sizes of atoms we looked up the liquid or solid density, ρ of the bulk element, which is the weight in grams of one cubic cm. We use Avagadro's number and the atomic mass to infer the volume per atom, and take the atomic size to be the size of a cube with that same volume:

$$D^3_{atom} = (1\ \text{cm})^3 \times \frac{\rho}{A} \times \frac{1}{N_A}. \quad (1)$$

Table 1: Sizes of Atoms

Element	Atomic Mass	Density (g/cc)	Atomic Size (nm)
Hydrogen (Liq.)	1	0.07	.29
Helium (Liq.)	4	0.125	.38
Water	18	1.	.22 per atom
Copper (sol.)	64	8.9	.24

Now this is a genuine surprise. More or less, all atoms seem to be the same size! We will leave explaining this as a puzzle for the reader. Notice that in the graph atoms lie almost in a vertical line showing one size but a range of masses.

The radii of nuclei have been measured, and they do not behave as strangely as atoms do. Their radius is proportional to the cube root of the number of particles they are made from, just as though they were a bunch of snowballs packed into one bigger ball. They lie along a sloped line corresponding to larger sizes for heavier nuclei.

Finding patterns: Notice that really a lot of objects lie along a straight line joining a hydrogen atom to the sun. This line corresponds to a constant density, the density of water.

$$M_{obj} = 1000\frac{\mathrm{kg}}{\mathrm{m}^3} \times D^3 \tag{2}$$

Molecules, people and the sun all have the same density, the density of close-packed atoms. Yet the pressures in these objects are quite different. I take from this that atoms are **very incompressible**. Atomic size is determined by a tradeoff between the electrical Coulomb attraction and the quantum mechanical energy cost that making a small object requires a compact wave function. These two balanced forces must involve a lot more energy than is available from the pressure and kinetic energy of atoms in typical places, even at the centre of the sun.

There is the start of a second constant density line, running from a single proton up to a neutron star. This line corresponds to nuclear density. But we have only put stuff at both ends of this line, and not in the middle. It seems nowhere near as interesting as the atomic density line. Why aren't there nucleii of mass 2000? Objects with the mass of a horse and the size of a bacterium? If nuclear forces were long range, all these objects would be stable, and our universe would be a lot different than it is! Again, we leave the details of the size the largest nucleus and and the smallest neutron star as puzzles to the reader.

There are two lines put onto this graph in addition to the constant density lines.

The gravitational potential energy of a small particle of mass m a distance R from a particle whose mass is M is

$$E_{grav} = -\frac{GMm}{R}. \tag{3}$$

If the ratio of M/R is large enough this potential is larger than the full rest mass energy of the small particle, mc^2 and the particle can not escape, or even send light out. Thus, whenever

$$\frac{GM}{Rc^2} \geq 1 \tag{4}$$

we have a black hole. The value of the radius for which Eq 4 equals unity is called the Schwarzschild radius, R_S. Notice that large black holes do not need to be particularly *dense*. An object with the density of air, but the mass of a galaxy would be a black hole. However, it is pretty difficult to imagine how to keep such an object from collapsing within its own horizon and becoming very dense!

There are objects on or near the line defined by Eq 4, but none well above it. This is because we can not see into that region. Objects above the line appear to have size R_S.

The fourth line on the graph equates the rest energy of an object, $E = mc^2$ to the wavelength of a photon of light with the same energy $E = h\nu = hc/\lambda$ so

$$mc = \frac{h}{\lambda_C}. \tag{5}$$

The wavelength which makes this equality hold is called the Compton wavelength.

Notice that neutrons and protons sit exactly at the intersection of this line and the line of nuclear density! If these particles were massless shiny boxes full of light the wavelength of the light would have to fit inside, and they would have just the size-mass relation that they do have. This is the modern view of nucleon structure, that the proton is a composite particle composed of much lighter objects confined to a small volume. Quantum mechanics and the strong nuclear forces both play a role in defining nucleon size.

So we see that gravity, electrical forces, nuclear forces and quantum mechanics all play well defined roles in determining the sizes of stable objects, roles which become clear without even solving any detailed problems associated with any particular objects.

Concluding puzzle: There is one last spot on the graph to notice. It is a place where we have not put any objects at all, but the line for the black hole size, R_S, crosses the line for the Compton wavelength λ_C at a very small size but an almost macroscopic mass of 10μg. At this size and mass we expect *both* quantum mechanics and gravity to be important. The trouble is that there is no clear recipe for including gravity into quantum mechanics, or quantum mechanics into gravity at the moment. This is a funny situation. We might have no idea *what* goes on at the intersection of these two lines, but we know right *where* it goes on! Physics in this neighborhood is left as a final, and very difficult, puzzle to the reader.

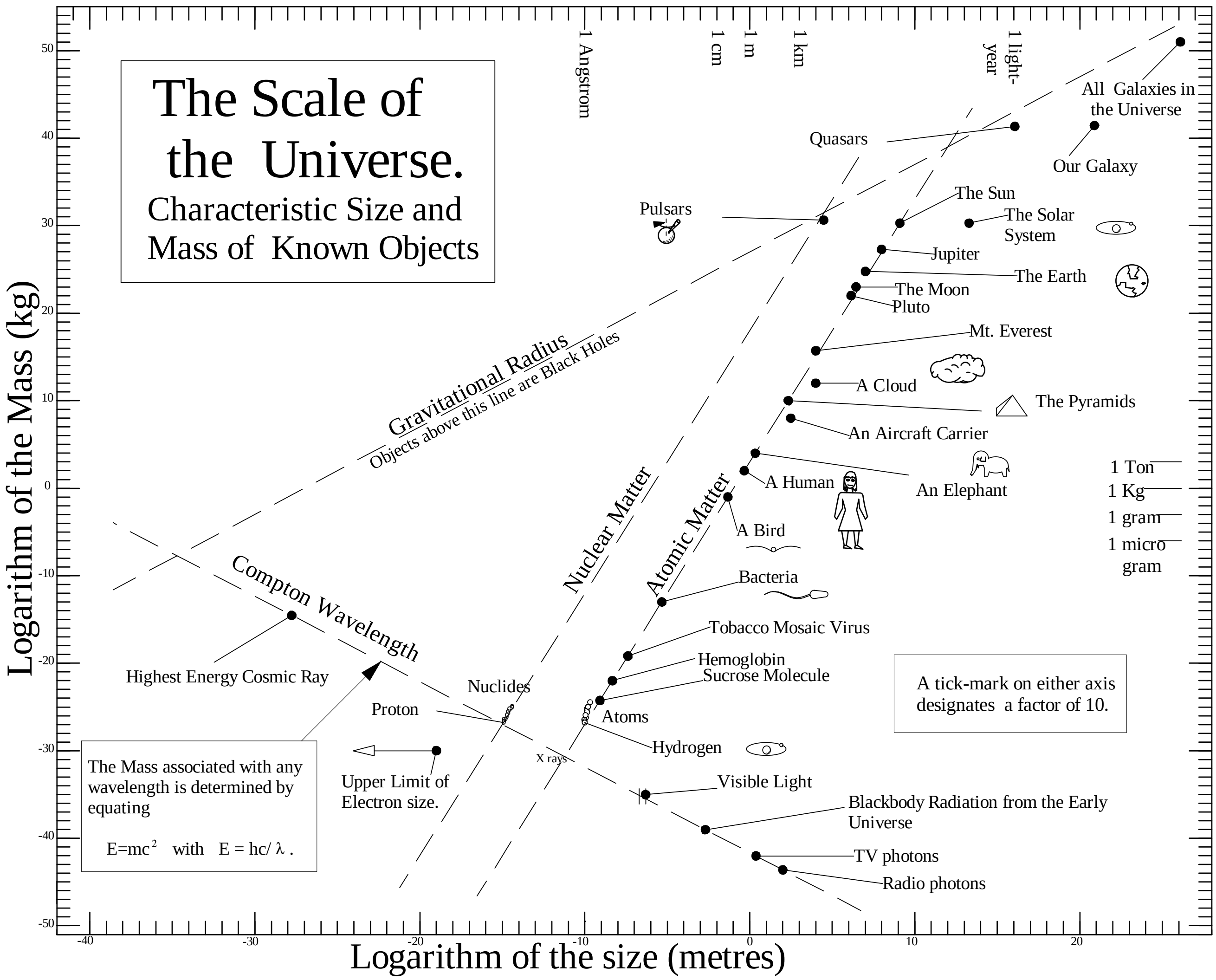

Figure 1: **Sizes and masses of selected objects:** Masses and linear sizes of common objects are shown as spots on this logarithmic graph which runs over 100 factors of ten vertically and 70 factors of ten horizontally. The large number of objects found near the line of constant density running from near a single hydrogen atom up past the sun shows how incompressible atoms are over a large range of pressure and temperature. The fact that a proton and a neutron star lie on a similar constant density line shows that nucleii are also incompressible, but the absence of objects in between the two ends of that line indicates that nucleii larger than a few hundred particles are not stable.