PHYS 304

MIDTERM 1 SOLUTION

FALL 2017

1. a) $$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = A^2 \int_{-\infty}^{\infty} (\psi_0 + i\sqrt{3}\psi_2)(\psi_0 - i\sqrt{3}\psi_2)\, dx$$

$$= A^2 \left[\int_{-\infty}^{\infty} |\psi_0|^2 dx + 3 \int_{-\infty}^{\infty} |\psi_2|^2 dx \right] = 4A^2$$

$$\boxed{A = \frac{1}{2}}$$

b) $$\Psi(x,t) = \frac{1}{2}\left[\psi_0(x)\, e^{-iE_0 t/\hbar} - i\sqrt{3}\,\psi_2(x)\, e^{-iE_2 t/\hbar} \right], \quad E_n = \hbar\omega(n + \tfrac{1}{2})$$

$$|\Psi(x,t)|^2 = \frac{1}{4}\Big[|\psi_0(x)|^2 + 3|\psi_2(x)|^2 - i\sqrt{3}\,\psi_0^*(x)\psi_2(x)\, e^{i(E_2-E_1)t/\hbar} + i\sqrt{3}\,\psi_0(x)\psi_2^*(x)\, e^{-i(E_2-E_1)t/\hbar} \Big]$$

$$= \frac{1}{4}\left[\psi_0^2 + 3\psi_2^2 + 2\sqrt{3}\,\psi_0\psi_2 \sin\frac{(E_2-E_1)t}{\hbar} \right]$$

c) This is NOT a stationary state because $|\Psi(x,t)|^2$ changes with time.

d) $$\Psi(x,0) = c_0\psi_0 + c_2\psi_2; \qquad c_0 = \frac{1}{2}, \quad c_2 = \frac{-i\sqrt{3}}{2}$$

Energy measurement on the state Ψ will produce
$E_0 = \frac{1}{2}\hbar\omega$ with probability $P_0 = |c_0|^2 = \frac{1}{4}$ or
$E_2 = \frac{5}{2}\hbar\omega$ with probability $P_2 = |c_2|^2 = \frac{3}{4}$.

e) $$\langle H \rangle = P_0 E_0 + P_2 E_2 = \hbar\omega\left(\frac{1}{2}\cdot\frac{1}{4} + \frac{5}{2}\cdot\frac{3}{4}\right) = 2\hbar\omega$$

2. Agree. Wavefunctions Ψ that satisfy $\hat{H}\Psi = E\Psi$ have the time evolution $\Psi(x,t) = e^{-iEt/\hbar}\,\Psi(x,0)$ and therefore $|\Psi(x,t)|^2 = |\Psi(x,0)|^2$ is time independent $\Rightarrow$ Stationary state.

3. a)

$$\psi(x) = \begin{cases} A e^{ikx} + B e^{-ikx}, & x < 0 \\ F e^{ilx}, & x > 0 \end{cases} \qquad k = \frac{\sqrt{2mE}}{\hbar}, \quad l = \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

A - incident wave, B - reflected wave, F - transmitted wave

b) Boundary conditions at $x=0$:

(i) $\psi(0^-) = \psi(0^+) \Rightarrow A + B = F$

(ii) $\psi'(0^-) = \psi'(0^+) \Rightarrow ik(A-B) = ilF$

Eliminate F:

$$\frac{A+B}{A-B} = \frac{k}{l} \rightarrow \frac{B}{A} = \frac{\frac{k}{l}-1}{\frac{k}{l}+1}$$

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{k-l}{k+l}\right)^2 = \boxed{\left(\frac{\sqrt{E+V_0}-\sqrt{E}}{\sqrt{E+V_0}+\sqrt{E}}\right)^2}$$

For $E = V_0$,

$$\boxed{R = \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 \simeq 0.029}$$