# 1 (10 points)

## 1.1

Instead of deriving an equation of motion for the Green's function (which we have done in class), we simply express the $u_q$ in terms of creation and annihilation operators $a_q$, $a_q^\dagger$:

$$D_q^0(t) = -i\langle \mathcal{T} u_q(t) u_{-q}(0)\rangle = -\frac{i}{2M\Omega_q}\left[\theta(t)\langle\left(a_q(t)+a_{-q}^\dagger(t)\right)\left(a_{-q}+a_q^\dagger\right)\rangle + \theta(-t)\langle\left(a_{-q}+a_q^\dagger\right)\left(a_q(t)+a_{-q}^\dagger(t)\right)\rangle\right]$$

From the Heisenberg equation of motion, we have $a_q(t) = e^{-i\Omega_q} a_q(0)$ and $a_q^\dagger(t) = (a_q(t))^\dagger$. Inserting this and realizing that $a_q|0\rangle = 0$ and $\langle 0| a_q^\dagger = 0$, we get

$$D_q^0(t) = -\frac{i}{2M\Omega_q}\left[\theta(t)e^{-i\Omega_q t}\langle a_q a_q^\dagger\rangle + \theta(-t)e^{i\Omega_q t}\langle a_{-q}a_{-q}^\dagger\rangle\right] = -\frac{i}{2M\Omega_q}\left[\theta(t)e^{-i\Omega_q t} + \theta(-t)e^{i\Omega_q t}\right]$$

Here we have made the assumption that $\Omega_q = \Omega_{-q}$.

Then,

$$\begin{aligned} D_q(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} D_q^0(t) &= -\frac{i}{2M\Omega_q}\left[\int_0^\infty dt e^{-i\Omega_q t} e^{i(\omega+i\eta)t} + \int_{-\infty}^0 dt e^{i\Omega_q t} e^{i(\omega-i\eta)t}\right] \\ &= \frac{1}{2M\Omega_q}\left[\frac{1}{\omega-\Omega_q+i\eta} - \frac{1}{\omega+\Omega_q-i\eta}\right] = \frac{1}{2M\Omega_q}\frac{2\Omega_q+2i\eta}{\omega^2-\Omega_q^2+\eta^2+2\Omega_q i\eta} \\ &= \frac{1}{M}\frac{1}{\omega^2-\Omega_q^2+i\tilde{\eta}} \end{aligned}$$

where $\tilde{\eta} = 2\Omega_q\eta$ and we haved used that $2i\eta$ in the numerator vanishes in the implied limit of $\eta \to 0$ without causing any divergence.

## 1.2

The Matsubara (imaginary time) Green's function is defined as

$$\mathcal{D}_q^0(\tau) = \langle \mathcal{T} u_q(\tau) u_{-q}(0)\rangle$$

We will compute it for the phonon Hamiltonian $H_{ph} = \sum_q \Omega_q a_q^\dagger a_q$ using two different ways: by finding its *equation of motion* and by *explicit evaluation* of the thermal expectation value.

For the EOM approach, the basic strategy is to take time derivates until we obtain a closed set of differential equations. We start with

$$\partial_\tau \mathcal{D}_q^0(\tau) = \delta(\tau)\langle[u_q(0), u_{-q}(0)]\rangle + \langle \mathcal{T}\partial_\tau u_q(\tau) u_{-q}\rangle$$

Here, the commutator turns out to be zero,

$$[u_q(0), u_{-q}(0)] = \frac{1}{2M\Omega_q}\left[a_q + a_{-q}^\dagger, a_{-q} + a_q^\dagger\right] = \frac{1}{2M\Omega_q}\left(\left[a_q, a_q^\dagger\right] + \left[a_{-q}^\dagger, a_{-q}\right]\right) = 0$$

Since $a_q(\tau) = e^{\tau H_{ph}} a_q(0) e^{-\tau H_{ph}}$ it follows that $\partial_\tau a_q(\tau) = [H_{ph}, a_q(\tau)] = \sum_k \Omega_k \left[a_k^\dagger a_k, a_q(\tau)\right] = -\Omega_q a_q(\tau)$ and therefore $a_q(\tau) = e^{-\Omega_q \tau} a_q(0)$. An analogous calculation yields $a_q^\dagger(\tau) = e^{\Omega_q \tau} a_q^\dagger(0)$. Note that in imaginary time, $a_q^\dagger(\tau) \neq (a_q(\tau))^\dagger$! Take a moment to think why this is not the case.

Making use of the explicit form of the time dependence, we get

$$\partial_\tau \mathcal{D}_q^0(\tau) = \frac{1}{\sqrt{2M\Omega_q}} \langle \mathcal{T} \left( -\Omega_q a_q(\tau) + \Omega_q a_{-q}^\dagger(\tau) \right) u_{-q} \rangle$$

This still does not help. So let's take another time-derivative.

$$\partial_\tau^2 \mathcal{D}_q^0(\tau) = \delta(\tau) \frac{1}{2M} \langle \left[ -a_q + a_{-q}^\dagger, a_{-q} + a_q^\dagger \right] \rangle + \Omega_q^2 \langle \mathcal{T} u_q(\tau) u_{-q} \rangle = -\frac{1}{M}\delta(\tau) + \Omega_q^2 \mathcal{D}_q^0(\tau)$$

We write the Green's function as a Fourier series $\mathcal{D}_q^0(\tau) = \sum_n e^{i\omega_n \tau} \mathcal{D}_q^0(\omega_n)$, where $\omega_n$ are bosonic Matsubara frequencies $\omega_n = 2\pi n/\beta$. The $\delta(\tau)$-function can be expressed as $\delta(t) = \frac{1}{\beta}\sum_n e^{i\omega_n \tau}$. Inserting all of this gives

$$\left(-\omega_n^2 - \Omega_q^2\right) \mathcal{D}_q^0(\omega_n) = -\frac{1}{\beta M}$$

so that the final result is

$$\mathcal{D}_q^0(\omega_n) = \frac{1}{\beta M} \frac{1}{\omega_n^2 + \Omega_q^2} .$$

Alternatively, we can do the following:

$$\begin{aligned} 2M\Omega_q \mathcal{D}_q^0(\tau) &= \theta(\tau) \langle \left( a_q e^{-\Omega_q \tau} + a_{-q}^\dagger e^{\Omega_q \tau} \right) \left( a_{-q} + a_q^\dagger \right) \rangle + \theta(-\tau) \langle \left( a_{-q} + a_q^\dagger \right) \left( a_q e^{-\Omega_q \tau} + a_{-q}^\dagger e^{\Omega_q \tau} \right) \rangle \\ &= \theta(\tau) \left[ e^{-\Omega_q \tau} \langle a_q a_q^\dagger \rangle + e^{\Omega_q \tau} \langle a_{-q}^\dagger a_{-q} \rangle \right] + \theta(-\tau) \left[ e^{-\Omega_q \tau} \langle a_q^\dagger a_q \rangle + e^{\Omega_q \tau} \langle a_{-q} a_{-q}^\dagger \rangle \right] \end{aligned}$$

Now we make use of the fact that $\langle a_q^\dagger a_q \rangle = n_q = \frac{1}{e^{\beta \Omega_q} - 1}$.

$$\begin{aligned} 2M\Omega_q \mathcal{D}_q^0(\tau) &= \theta(\tau) \left[ e^{-\Omega_q \tau} (n_q + 1) + e^{\Omega_q \tau} n_q \right] + \theta(-\tau) \left[ e^{-\Omega_q \tau} n_q + e^{\Omega_q \tau} (n_q + 1) \right] \\ &= e^{-\Omega_q |\tau|} (n_q + 1) + e^{\Omega_q |\tau|} n_q \end{aligned}$$

And finally we perform the Fourier series

$$\begin{aligned} \mathcal{D}_q^0(\omega_n) &= \frac{1}{2M\Omega_q \beta} \int_0^\beta d\tau e^{-i\omega_n \tau} \left[ e^{-\Omega_q \tau} (n_q + 1) + e^{\Omega_q \tau} n_q \right] \\ &= \frac{1}{2M\Omega_q \beta} \left[ \frac{e^{-\Omega_q \beta}}{-i\omega_n - \Omega_q} (n_q + 1) + \frac{e^{\Omega_q \beta}}{-i\omega_n + \Omega_q} n_q - \frac{1}{-i\omega_n - \Omega_q} (n_q + 1) - \frac{1}{-i\omega_n + \Omega_q} n_q \right] \\ &= \frac{1}{2M\Omega_q \beta} \left[ \frac{1}{i\omega_n + \Omega_q} - \frac{1}{i\omega_n - \Omega_q} \right] = \frac{1}{M\beta} \frac{1}{\omega_n^2 + \Omega_q^2} \end{aligned}$$

## 1.3

We obtain the retarded phonon Green's function using

$$D_q^R(\omega) = -\beta \mathcal{D}_q^0(\omega_n \to -i\omega + \eta) = \frac{1}{2M\Omega_q} \left( \frac{1}{\omega - \Omega_q + i\eta} - \frac{1}{\omega + \Omega_k + i\eta} \right)$$

The spectral function is

$$\begin{aligned} A_q(\omega) &= -\frac{2}{1 - e^{-\beta\omega}} \text{Im} D_q^R(\omega) = \frac{2\pi}{1 - e^{-\beta\omega}} \frac{1}{2M\Omega_q} \left( \delta(\omega - \Omega_q) - \delta(\omega + \Omega_q) \right) \\ &= \frac{\pi}{M\Omega_q} \left[ (n_q + 1)\delta(\omega - \Omega_q) + n_q \delta(\omega + \Omega_q) \right] \end{aligned}$$

# 2 (20 points)

## 2.1

The Feynman-Dyson expansion to $n$-th order is given by

$$G_k(t) = -i\sum_{n=0}^{\infty}\frac{(-i\Lambda)^n}{n!}\sum_{q_1,k_1}\int_{-\infty}^{\infty}dt_1\cdots\sum_{q_n k_n}\int_{-\infty}^{\infty}dt_n\,\langle\Phi_0|\,\mathcal{T}c_k(t)c_k^\dagger(0)v_{-q_1}(t_1)c^\dagger_{k_1-q_1}(t_1)c_{k_1}(t_1)$$
$$\times\cdots\times v_{-q_n}(t_n)c^\dagger_{k_n-q_n}(t_n)c_{k_n}(t_n)\,|\Phi_0\rangle$$

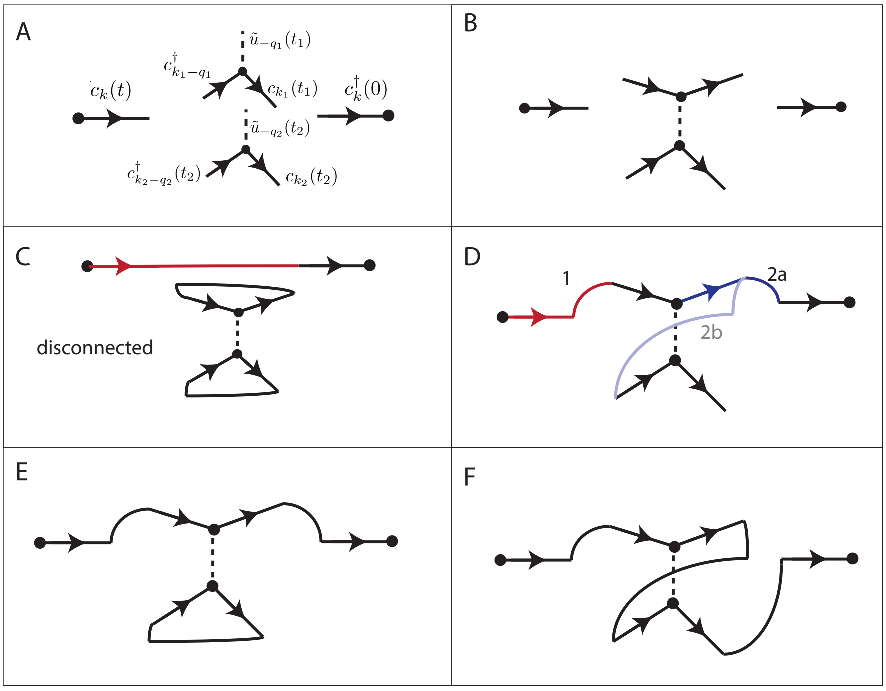
where we have defined $v_q = \sqrt{2M\Omega_q}u_q = a_q + a^\dagger_{-q}$ and used that the perturbation Hamiltonian is $H_1 = \Lambda\sum_{qk} v_{-q}c^\dagger_{k-q}c_k$.

Up to second order, the expression is

$$G_k(t) = G^0(t) - i\frac{(-i\Lambda)^2}{2}\sum_{q_1q_2k_1k_2}\int_{-\infty}^{\infty}dt_1dt_2\,\langle\Phi_0|\,\mathcal{T}c_k(t)c_k^\dagger(0)v_{-q_1}(t_1)c^\dagger_{k_1-q_1}(t_1)c_{k_1}(t_1)v_{-q_2}(t_2)c^\dagger_{k_2-q_2}(t_2)c_{k_2}(t_2)\,|\Phi_0\rangle \quad (1)$$

We now want to use Wick's theorem to simplify the expectation value. Since there are 8 operators in the expectation value, there is a lot of possible contractions. However, most of them vanish, and some give the same result.

It is best to organize all contraction using diagrams:




In panel A, we have represented each operator by a line. Fermion operators $c$ are represented by a full line. An arrow leading out of the black dot represent an annihilation operator; an arrow leading into a dot represents a creation operator. Since expectation values of to creation operators vanish, we can only pair $c$ with $c^\dagger$. In the arrow notation it means that we can only connect two legs with the same arrow flow.

Phonon operators $u$ are represented by a dashed line. Since there are only two $u$ operators, we have no choice other than contracting them which leads to the diagram in panel B.

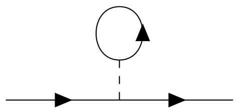
We could also be tempted to pair $c_k(t)$ and $c_k^\dagger(0)$. But this leads to a disconnected diagram, as shown in panel C. Disconnected diagrams do not contribute to the Green's function as they are cancelled by the denominator (DS 3.1.16) that we are omitting.

Now consider the red leg in panel D. We have two options to connect it. Further inspection reveals that both options yield the same result, since the interaction vertex is symmetric and all of its momentum variables are summed over. So we can safely make a choice, and remember that we need to multiply any resulting diagram by a factor of 2.

Now consider the blue leg in panel D. We have two distinct choices to connect it. The first one is shown by a dark blue line. The remaining two fermion legs need to be connected, yielding the diagram in panel E. It corresponds to the contraction

$$\langle\Phi_0|\,\mathcal{T}c_k(t)c_k^\dagger(0)v_{-q_1}(t_1)c_{k_1-q_1}^\dagger(t_1)c_{k_1}(t_1)v_{-q_2}(t_2)c_{k_2-q_2}^\dagger(t_2)c_{k_2}(t_2)\,|\Phi_0\rangle \tag{2}$$

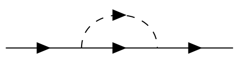
We can clean up the diagram so that is looks like this:



The second option (2b) shown in panel D yields the diagram in panel F. It corresponds to the following contraction:

$$\langle\Phi_0|\,\mathcal{T}c_k(t)c_k^\dagger(0)v_{-q_1}(t_1)c_{k_1-q_1}^\dagger(t_1)c_{k_1}(t_1)v_{-q_2}(t_2)c_{k_2-q_2}^\dagger(t_2)c_{k_2}(t_2)\,|\Phi_0\rangle \tag{3}$$

In clean form, the corresponding diagram look like this:



Let us now perform the contraction in Eq. (3) explicitly:

$$\begin{aligned}
&\langle\Phi_0|\,\mathcal{T}c_k(t)c_k^\dagger(0)v_{-q_1}(t_1)c_{k_1-q_1}^\dagger(t_1)c_{k_1}(t_1)v_{-q_2}(t_2)c_{k_2-q_2}^\dagger(t_2)c_{k_2}(t_2)\,|\Phi_0\rangle \\
= &\langle\Phi_0|\,\mathcal{T}c_k(t)c_{k_1-q_1}^\dagger(t_1)\,|\Phi_0\rangle\,\langle\Phi_0|\,\mathcal{T}c_{k_2}(t_2)c_k^\dagger(0)\,|\Phi_0\rangle\,\langle\Phi_0|\,\mathcal{T}v_{-q_1}(t_1)v_{-q_2}(t_2)\,|\Phi_0\rangle\,\langle\Phi_0|\,\mathcal{T}c_{k_1}(t_1)c_{k_2-q_2}^\dagger(t_2)\,|\Phi_0\rangle \\
= &G_k^0(t-t_1)\delta(k-k_1+q_1)G_k^0(t_2)\delta(k_2-k)D_{-q_1}^0(t_1-t_2)\delta(q_1+q_2)G_{k_1}^0(t_1-t_2)\delta(k_1-k_2+q_2)2M\Omega_{q_1}
\end{aligned}$$

For the third line, we have applied the definition of the time-ordered real-time Green's functions $G^0$ and $D^0$. Note that one has to keep track of minus signs that arise when two fermionic operators are commuted past each other. In the present case, there is no overall minus sign. We now insert this into Eq. (1):

$$\begin{aligned}
G_k(t) &= G^0(t) + 2\frac{i\Lambda^2}{2}\sum_q\int_{-\infty}^{\infty} dt_1dt_2 G_k^0(t-t_1)D_q^0(t_1-t_2)G_{k-q}^0(t_1-t_2)G_k^0(t_2-0)2M\Omega_q \\
&= G^0(t) + 2\frac{i\Lambda^2}{2}\sum_q\int_{-\infty}^{\infty} dt_1dt_2\frac{1}{(2\pi)^4}\int d\omega_1d\omega_2d\omega_3d\omega_4e^{-i\omega_1(t-t_1)}e^{-i(\omega_2+\omega_3)(t_1-t_2)}e^{-i\omega_4t_2} \\
&\quad \times G_k^0(\omega_1)D_q^0(\omega_2)G_{k-q}^0(\omega_3)G_k^0(\omega_4)2M\Omega_q \\
&= G^0(t) + 2\frac{i\Lambda^2}{2}\sum_q\frac{1}{(2\pi)^2}\int d\omega d\omega' e^{-i\omega t}G_k^0(\omega)D_q^0(\omega')G_{k-q}^0(\omega-\omega')G_k^0(\omega)2M\Omega_q
\end{aligned}$$

## 2.2

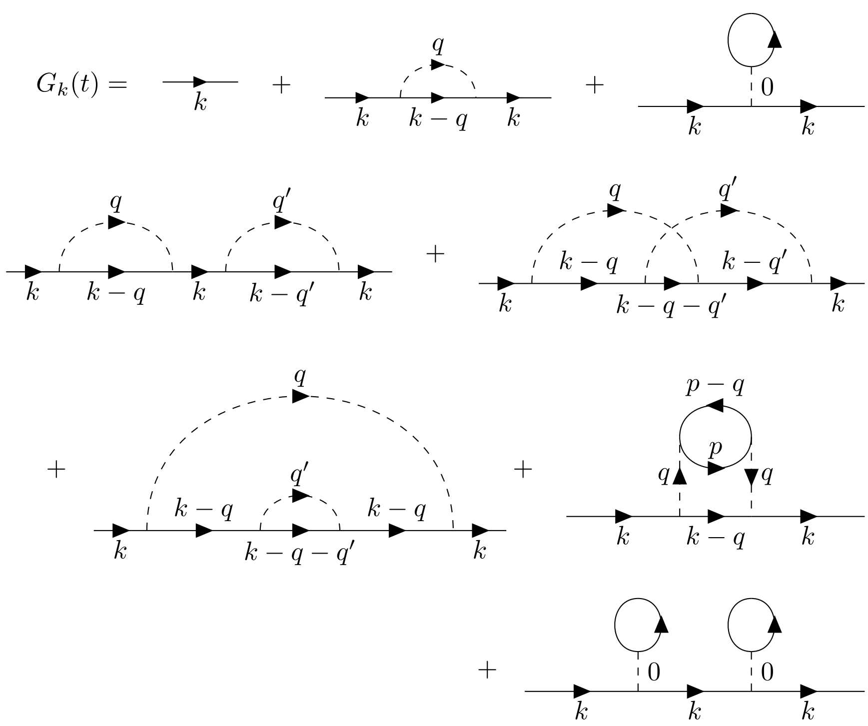
We perform the Fourier-transform:

$$\begin{aligned} G_k(\omega) &= G^0(\omega) + 2\frac{i\Lambda^2}{2}\sum_q \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' G_k^0(\omega) D_q^0(\omega') G_{k-q}^0(\omega-\omega') G_k^0(\omega) 2M\Omega_q \\ &= G^0(\omega) + G_k^0(\omega)\Sigma_k(\omega) G_k^0(\omega) \end{aligned}$$

where

$$\Sigma_k(\omega) = \frac{i\Lambda^2 M}{\pi}\sum_q \Omega_q \int_{-\infty}^{\infty} d\omega' D_q^0(\omega') G_{k-q}^0(\omega-\omega')\,.$$

For your entertainment, you can try to find all possible diagrams up to 4th order. Below are a few of these diagrams. All in all, there are 2 topologically distinct first order diagrams, and 10 distinct one at second order. The full list of diagrams is given in DS Eq. (6.3.13). In principle, one can follow above derivation to get the integral expression for each of these diagrams. However, it is possible, and maybe a bit more convenient, to directly translate a diagram into an integral expression by following the so called *Feynman rules*.

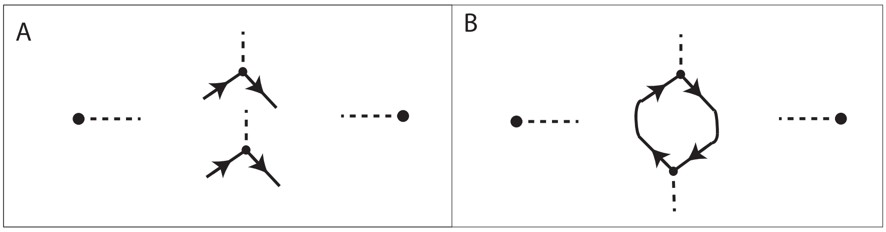



## 2.3

The expression for the phonon propagator to second order is

$$D_q(t) = D_q^0(t) - i\frac{(-i\Lambda)^2}{2}\sum_{q_1 q_2 k_1 k_2}\int_{-\infty}^{\infty} dt_1 dt_2 \langle \mathcal{T} u_q(t) u_{-q}(0) v_{-q_1}(t_1) c_{k_1-q_1}^\dagger(t_1) c_{k_1}(t_1) v_{-q_2}(t_2) c_{k_2-q_2}^\dagger(t_2) c_{k_2}(t_2)\rangle$$

We again organize the possible Wick contractions using a diagram:

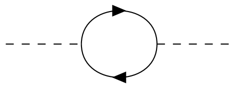



Panel A is similar to what we had before. The difference is that the outer legs are now phonon-legs (dashed line) since we are computing the phonon propagator. There is only one way to connect the 4 fermionic legs, see panel B. After that, there are two ways of pairing the phonon legs, but again these two pairings are equivalent, so everything results in a factor of 2.

Explicitly, the contraction is:

$$\langle \mathcal{T} u_q(t) u_{-q}(0) v_{-q_1}(t_1) c^\dagger_{k_1-q_1}(t_1) c_{k_1}(t_1) v_{-q_2}(t_2) c^\dagger_{k_2-q_2}(t_2) c_{k_2}(t_2)\rangle$$
$$= -2M\Omega_q D^0_q(t-t_1)\delta(q-q_1) G^0_{k_1-q_1}(t_2-t_1)\delta(k_1-q_1-k_2) G^0_{k_1}(t_1-t_2)\delta(k_1-k_2+q_2) D^0_q(t_2)\delta(q-q_2)$$

Note that one has to keep track of minus signs that arise when commuting to fermionic operators! In this case, we get a total minus sign! The above expression corresponds to the following diagram:



Inserting this into Eq. (4) yields

$$D_q(t) = D^0_q(t) - 2i\frac{\Lambda^2}{2}\sum_k \int_{-\infty}^{\infty} dt_1 dt_2 2M\Omega_q D^0_q(t-t_1) G^0_{k-q}(t_2-t_1) G^0_k(t_1-t_2) D^0_q(t_2-0)$$

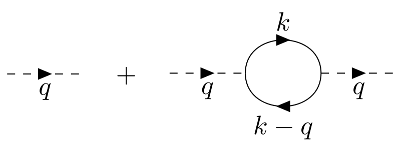
Fourier transforming just like before gives

$$\begin{aligned} D_q(\omega) &= D^0_q(\omega) - 2i\frac{1}{2\pi}\frac{\Lambda^2}{2}\sum_k \int_{-\infty}^{\infty} d\omega' 2M\Omega_q D^0_q(\omega) G^0_{k-q}(\omega'-\omega) G^0_k(\omega') D^0_q(\omega) \\ &= D^0_q(\omega) + D^0_q(\omega)\Pi_q(\omega) D^0_q(\omega) \end{aligned}$$

where

$$\Pi_q(\omega) = -2i\frac{1}{2\pi}\frac{\Lambda^2}{2}\sum_k \int_{-\infty}^{\infty} d\omega' 2M\Omega_q G^0_{k-q}(\omega'-\omega) G^0_k(\omega')$$

The Feynman-Dyson expansion of the full phonon propagator up to second order has the following diagrammatic representation:

$D_q(t) =$ 

# 3 (10 points)

## 3.1

For finite temperature, the above calculation remains the same apart from some factors of $i$ and integration boundaries at 0 and $\beta$ instead of $-\infty$and $\infty$. The expression for the phonon propagator to second order is

$$\mathcal{D}_q(\tau) = \mathcal{D}_q^0(\tau) + \frac{\Lambda^2}{2} \sum_{q_1 q_2 k_1 k_2} \int_0^\beta d\tau_1 d\tau_2 \langle \mathcal{T} u_q(\tau) u_{q-}(0) v_{-q_1}(\tau_1) c^\dagger_{k_1-q_1}(\tau_1) c_{k_1}(\tau_1) v_{-q_2}(\tau_2) c^\dagger_{k_2-q_2}(\tau_2) c_{k_2}(\tau_2) \rangle_\beta$$

Inserting this into Eq. (4) yields

$$\mathcal{D}_q(\tau) = \mathcal{D}_q^0(\tau) - 2\frac{\Lambda^2}{2} \sum_k \int_0^\beta d\tau_1 d\tau_2 2M\Omega_q \mathcal{D}_q^0(\tau - \tau_1) \mathcal{G}^0_{k-q}(\tau_2 - \tau_1) \mathcal{G}^0_k(\tau_1 - \tau_2) \mathcal{D}_q^0(\tau_2 - 0)$$

Applying the definition of the Fourier series for Matsubara frequencies

$$\begin{aligned}
\mathcal{G}^0(\tau) &= \sum_n e^{i\omega_n \tau} \mathcal{G}^0(\omega_n) \\
\mathcal{G}^0(\omega_n) &= \frac{1}{\beta} \int_0^\beta e^{-i\omega_n \tau} \mathcal{G}^0(\tau) \\
\int_0^\beta e^{i(\omega_n - \omega_m)\tau} d\tau &= \beta \delta_{n,m}
\end{aligned}$$

gives the phonon propagator in frequency space

$$\begin{aligned}
\mathcal{D}_q(\omega_n) &= \mathcal{D}_q^0(\omega_n) - \Lambda^2 \beta^2 \sum_k \sum_{\omega_m} 2M\Omega_q \mathcal{D}_q^0(\omega_n) \mathcal{G}^0_{k-q}(\omega_m - \omega_n) \mathcal{G}^0_k(\omega_m) \mathcal{D}_q^0(\omega_n) \\
&= \mathcal{D}_q^0(\omega_n) + \mathcal{D}_q^0(\omega_n) \Pi_q(\omega_n) \mathcal{D}_q^0(\omega_n)
\end{aligned}$$

where we have defined the phonon-self-energy

$$\Pi_q(\omega_n) = -2M\Lambda^2\beta^2\Omega_q \sum_k \sum_{\omega_m} \mathcal{G}^0_{k-q}(\omega_m - \omega_n) \mathcal{G}^0_k(\omega_m)$$

## 3.2

The fermionic Matsubara Green's function is given by

$$\mathcal{G}^0_k(\omega_n) = \frac{1}{\beta} \frac{1}{i\omega_n + \varepsilon_k}$$

We evaluate the Matsubara sum using the method of contour integration.

$$\begin{aligned}
C_q(\omega_n) &= \sum_{\omega_m} \mathcal{G}^0_{k-q}(\omega_m - \omega_n) \mathcal{G}^0_k(\omega_m) = -\frac{\beta}{2\pi i} \oint \frac{1}{e^{\beta\omega} + 1} \mathcal{G}^0_{k-q}(\omega_n - i\omega) \mathcal{G}^0_k(i\omega) \\
&= -\frac{1}{2\pi i \beta} \oint \frac{1}{e^{\beta\omega} + 1} \frac{1}{-i\omega_n - \omega + \varepsilon_{k-q}} \frac{1}{-\omega + \varepsilon_k}
\end{aligned}$$

We see that there are two poles at $\omega = \varepsilon_k$ and $\omega = \varepsilon_{k-q} + i\omega_n$. Computing the residues and also considering the minus sign from the reversed orientation of the contour, we get

$$\begin{aligned}
C_q(\omega_n) &= \frac{1}{\beta} \left( -\frac{1}{e^{\beta\varepsilon_k} + 1} \frac{1}{-i\omega_n + \varepsilon_{k-q} - \varepsilon_k} - \frac{1}{e^{\beta\varepsilon_{k-q}} + 1} \frac{1}{i\omega_n - \varepsilon_{k-q} + \varepsilon_k} \right) \\
&= \frac{1}{\beta} \frac{1}{i\omega_n + \varepsilon_k - \varepsilon_{k-q}} (f_k - f_{k-q})
\end{aligned}$$

where $f_k = 1/(e^{\beta\epsilon_k}+1)$. Note that $\omega_n = 2\pi n/\beta$ is a bosonic frequency, as opposed to $\omega_m = (2n+1)\pi/\beta$ which is fermionic. Thus, $e^{-i\omega_n\beta} = 1$. Using this our self-energy becomes

$$\Pi_q(\omega_n) = -2M\Lambda^2\beta\Omega_q \sum_k \frac{f_k - f_{k-q}}{i\omega_n + \varepsilon_k - \varepsilon_{k-q}}$$